UNTIL THE BITTER END: ON PROSPECT THEORY IN A DYNAMIC CONTEXT

SEBASTIAN EBERT AND PHILIPP STRACK*

Many economic and financial decisions depend crucially on their timing. People decide when to invest in a project, when to liquidate assets, or when to stop gambling in a casino. We provide a general result on prospect theory decision makers who are unaware of the time-inconsistency induced by probability weighting. If a market offers a sufficiently rich set of investment strategies, then such naïve investors postpone their decisions until forever. We illustrate the drastic consequences of this "never stopping" result, and conclude that probability distortion in combination with naïvité leads to unrealistic predictions for a wide range of dynamic setups.

KEYWORDS: Behavioral Economics, Disposition Effect, Irreversible Investment, Prospect Theory, Skewness Preference, Time-Inconsistency. JEL: G02, D03, D81.

1. INTRODUCTION

While expected utility theory (EUT, Bernoulli 1738/1954, von Neumann and Morgenstern 1944) is the leading normative theory of decision making under risk, cumulative prospect theory (CPT, Kahneman and Tversky 1979, Tversky and Kahneman 1992) is the most prominent positive theory. EUT is well-studied in both static and dynamic settings, ranging from game theory over investment problems to institutional economics. In contrast, research on CPT with all its aspects has mostly focused on the static case so far. This paper derives a fundamental result on the dynamic investment and gambling behavior predicted by CPT. This result has immediate and strong consequences for a number of dynamic decision problems such as irreversible investment, casino gambling, or the disposition effect.

Usually, CPT is characterized by four features: First, outcomes are evaluated by a value function relative to a reference point which separates all outcomes into gains and losses. Second, utility is S-shaped, i.e., convex for losses and concave for gains. Third, probabilities are distorted by inverse-S-shaped probability weighting functions (one for gains and one

Ebert: Institute for Financial Economics and Statistics, and Bonn Graduate School of Economics, University of Bonn, Adenauerallee 24 - 42, D-53113 Bonn, Germany; Email: sebastianebert@uni-bonn.de. Strack: Microsoft Research New England, One Memorial Drive, Cambridge, MA 02142, USA; E-mail: philipp.strack@gmail.com.

*First version: February 15, 2012. This version: July 12, 2013. Acknowledgments: We thank Nicholas Barberis, Stefano DellaVigna, Urs Fischbacher, Paul Heidhues, Daniel Krähmer, Ulrike Malmendier, Matthew Rabin, Sven Rady, Peter Wakker, and Georg Weizsäcker for helpful comments and discussions on the topic. We further thank seminar participants at Berkeley, Bonn Micro, Bonn Finance, Cologne, DIW Berlin, EM-LYON, Konstanz, LMU Munich, Mannheim, Tilburg, VU Amsterdam, Warwick Business School, as well as participants of the 2012 European Winter Meeting of the Econometric Society, the 2012 Verein für Socialpolitik Annual Meeting, the 2012 European Economic Association Annual Congress, the 2012 DIW Behavioral Economics Workshop, and the 2012 Fundamentals of Risk and Uncertainty Meeting for their comments and suggestions. This paper is part of Strack's dissertation submitted to the Bonn Graduate School of Economics. The project was initiated while Ebert participated in the 2nd Yale Summer School on Behavioral Finance and when Strack was a visiting researcher at Yale. Financial support from the Hausdorff Center for Mathematics and the German Research Foundation (DGF) is gratefully acknowledged.

for losses). This implies that small probabilities associated with large gains or losses are overweighted. Fourth, losses loom larger than gains, which is referred to as loss aversion.

This paper derives results on CPT that are consequences of (potentially very little) probability weighting alone. Our only assumptions on the value function are monotonicity and a weak form of differentiability. The reference point may be arbitrary and change over time. If some probability weighting is considered to be a fundamental element of prospect theory, then so are the results in this paper. And if one abstracts from prospect theory's probability weighting feature as many papers do, then this paper has nothing to say—except for that it may serve as motivation for why one wants to abstract from probability weighting.

The assumption we impose on the probability weighting function, while being the single driving force of this paper's results, is rather mild. In particular, we propose a new weighting function which we call the benchmark weighting function. Our results apply to any version of CPT whose weighting function for gains lies above this benchmark weighting function for at least one probability; a similar condition is imposed on the weighting function for losses. All commonly used inverse-S-shaped weighting functions, but also S-shaped weighting functions, satisfy these conditions.

Our dynamic results can be traced back to a seemingly innocuous result that we call *skewness preference in the small*. At any wealth level, a CPT agent wants to take a sufficiently right-skewed binary risk which is arbitrarily small, even if it has negative expectation. We call such a risk *attractive* to the CPT agent. Therefore, a CPT agent can always be lured into gambling by offering an attractive risk. We show that such a risk may be small, but attractive risks may also be large. For the original parametrization of Tversky and Kahneman (1992) and when the decision maker's reference point is her status quo, for example, there exists an attractive risk of *any* size. This result comes as a consequence of a result in Azevedo and Gottlieb (2012) in combination with skewness preference in the small.

Our main result is a theorem on the dynamic investment and gambling behavior of a prospect theory agent who is naïve, i.e., unaware of his time-inconsistency. For a very large class of gambles we show that naïve CPT agents never stop gambling. Intuitively, at any point in time the agent reasons "If I lose just a little bit more, I will stop. And if I gain, I will continue." We show that such a stop-loss strategy results in a right-skewed gambling experience which is attractive due to our static result: skewness preference in the small. However, since the agent is time-inconsistent, once a loss has occurred, a new attractive skewed gambling strategy comes to mind, and thus he continues gambling.

Formally, we derive a stopping theorem naïve for prospect theory agents which holds for a large class of stochastic processes that model the gambling or investment opportunities. These processes include the geometric Brownian motion (which is the most common choice to model the price development of an asset) as well as the arithmetic Brownian motion which could, for example, model the accumulated gains of an agent gambling in a casino. Notably, the process may imply arbitrarily large expected loss per time, and thus the result also holds for highly unfavorable investments with a very negative expected value. Because of its generality, the result has strong and immediate implications for numerous prominent economic and financial decision problems.

We present an analytical solution to a continuous time, infinite horizon casino gambling model in the spirit of Barberis (2012). We show that naïve CPT agents gamble in a casino until the bitter end, i.e., until bankruptcy.

We then investigate the irreversible investment problem of Dixit and Pindyck (1994) for a naïve CPT investor. More generally than that we show that CPT agents will never exercise an American option even if it is profitable to exercise immediately.

Finally, our results imply that CPT with probability distortion cannot predict the disposition effect (Shefrin and Statman 1985) for naïve investors. We show that naïve prospect theory investors with probability distortion will sell neither losers nor winners at any time. This is especially striking as Henderson (2012) shows that, in an identical continuous time investment model without probability distortion, prospect theory can explain the disposition effect.

Our extreme results are robust to finite and/or discrete time spaces, as long as the stochastic process allows for a rich set of possible investment strategies. In complete markets, for example, our result will hold irrespective of the time space.

Note that the time-inconsistency discussed here arises naturally because of probability weighting as originally proposed by Kahneman and Tversky. Therefore, the question of how this time-inconsistency may be resolved is immediate, and needs to be addressed in any application that takes prospect theory to the dynamic context. While in this paper we analyze the situation of a naïve agent, Xu and Zhou (2013) derive the initially optimal stopping strategy for a sophisticated agent who has access to a commitment device allowing her to stick to her initial plan. The case of a sophisticated prospect theory agent without commitment is, to the best of our knowledge, not solved yet.

What do we learn from this paper? We show that probability distortion of prospect theory in combination with naïvité leads to absurd predictions for a wide range of dynamic setups (including the standard models of financial markets). There are two possible conclusions from this observation:

First, naïvité may be the wrong resolution for the time-inconsistency induced by probability weighting. Second, one could conclude that in dynamic situations prospect theory should be employed without probability distortion. This conclusion seems especially attractive as prospect theory without probability distortion has been extensively and successfully applied in explaining various empirical phenomena such as the disposition effect (e.g. Henderson

2012 and Barberis and Xiong 2009) or life insurance (Gottlieb 2013). Prospect theory without probability distortion may also be appealing from a theoretical point of view as it leads to time-consistent behavior, making the distinction between naïvité, sophistication, and commitment superfluous.

In Section 2 we define CPT preferences with probability distortion and introduce the benchmark weighting functions. In Section 3 we present our static result that CPT implies skewness preference in the small. Section 4 presents the "never stopping" result. Section 5 discusses the implications for CPT models of casino gambling, real-option investment behavior, and the disposition effect. Section 6 discusses robustness towards discrete and finite time spaces. Section 7 reviews our results and assumptions and points out venues for future research. Further results and proofs are relegated to the appendix.

2. PROSPECT THEORY PREFERENCES WITH (AT LEAST) LITTLE PROBABILITY WEIGHTING

We consider an agent with CPT preferences over real-valued random variables X. For simplicity, first consider a binary risk $L(p, b, a)$ that yields outcome b with probability $p \in (0, 1)$, and $a < b$ otherwise. A prospect theory agent evaluates binary risks relative to a reference point $r \in \mathbb{R}$ as

$$(1) \qquad CPT(L(p,b,a)) = \begin{cases} (1 - w^+(p))U(a) + w^+(p)U(b), & \text{if } r \leq a \\ w^-(1-p)U(a) + w^+(p)U(b), & \text{if } a < r \leq b \\ w^-(1-p)U(a) + (1 - w^-(1-p))U(b) & \text{if } b < r \end{cases}$$

with non-decreasing weighting functions $w^-, w^+ : [0, 1] \to [0, 1]$ with $w^+(0) = w^-(0) = 0$ and $w^+(1) = w^-(1) = 1$, and a continuous, strictly increasing value function $U : \mathbb{R} \to \mathbb{R}$ with $U(r) = 0$ that satisfy Assumptions 1 and 2 stated below. Later on we will adopt a more general version of CPT which also applies to random variables X with possibly continuous outcomes:

$$(2) \qquad CPT(X) = \int_{\mathbb{R}_+} w^+(\mathbb{P}(U(X) > y))\mathrm{d}y - \int_{\mathbb{R}_-} w^-(\mathbb{P}(U(X) < y))\mathrm{d}y.$$

For binary risks X, formula (2) reduces to formula (1), and for general discrete prospects it reduces to the well-known form of Tversky and Kahneman (1992). To understand the results of this paper it is sufficient to have formula (1) in mind. The following assumption on the prospect theory value function means that any kinks it may have are not too extreme, which in particular excludes infinite loss aversion.

ASSUMPTION 1 (Value function) *The value function U has finite left derivative $\partial_- U(x) = \lim_{z \nearrow x} \frac{U(x)-U(z)}{x-z}$ and finite right derivative $\partial_+ U(x) = \lim_{z \searrow x} \frac{U(z)-U(x)}{z-x}$ at every wealth level*

x. *Further,* $\lambda = \sup_{x\in\mathbb{R}} \frac{\partial_- U(x)}{\partial_+ U(x)} < \infty$ *exists.*

CPT preferences are defined on changes in wealth relative to the reference point r rather than on absolute wealth levels. Typical choices for r are the status quo of current wealth or some other benchmark. For example, when investing in a risky asset, r could be the return of a risk-free investment. Realizations x of X with $x < r$ are referred to as losses, and realizations $x \geq r$ are called gains. Our results hold for any $r \in \mathbb{R}$. In other words, the choice of the reference point is immaterial to our findings.

Note that, as U is strictly increasing, it is differentiable almost everywhere so that $\lambda \geq 1$. In many specifications of prospect theory, the additional assumption is made that U is differentiable everywhere except at the reference point such that $\lambda = \frac{\partial_- U(r)}{\partial_+ U(r)}$. It is further assumed that $\lambda > 1$ and that the reflection property

$$(3) \qquad U(x) = \begin{cases} u(x-r), & \text{if } x \geq r \\ -\lambda u\left(-(x-r)\right), & \text{if } x < r \end{cases}$$

holds for some function $u : \mathbb{R}_+ \longrightarrow \mathbb{R}_+$, where $U(r) = 0$ implies that $u(0) = 0$. $\lambda > 1$ then implies that losses loom larger than gains to the CPT agent; see Köbberling and Wakker (2005) for an analysis of the loss aversion index $\frac{\partial_- U(r)}{\partial_+ U(r)}$. We allow for value functions that may have a kink not only at the reference point, because this can be important for the evaluation of risks with kinked payoff profiles such as option contracts. The original choice for u by Tversky and Kahneman (1992) was power utility. Since several caveats have been pointed out for the power utility parametrization, exponential utility has become another popular choice (de Giorgi and Hens 2006).[1]

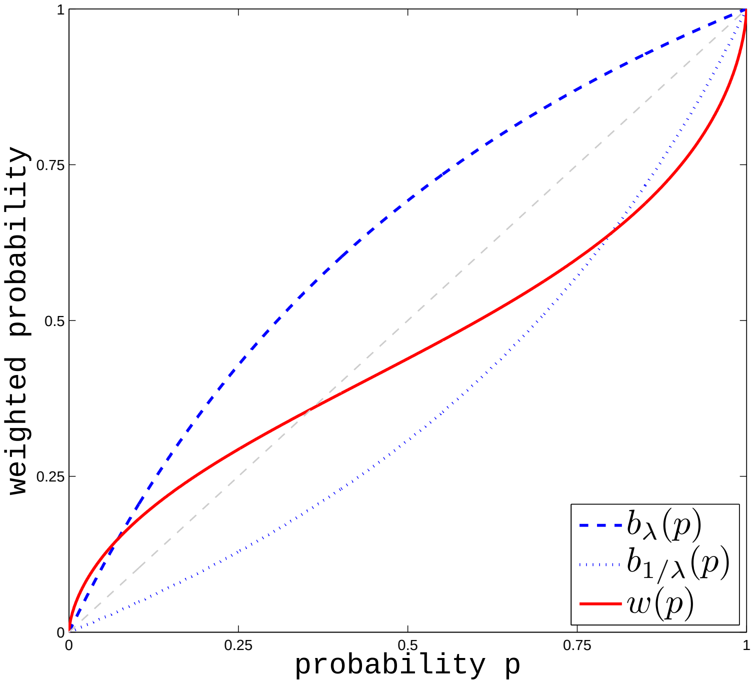
The final important feature of CPT is probability weighting. While probability weighting is the driving force of this paper's results, the actual assumptions we impose on the probability weighting functions are very mild. Our only requirement is that there exists one probability p which is overweighted at least a bit by w^+, and the counter-probability $1-p$ is underweighted by w^-. All commonly used weighting functions satisfy this condition for a small enough p. The rest of this section formalizes this important assumption of our paper and discusses it in detail.

For $\theta > 0$ we define the function family $b_\theta : [0,1] \longrightarrow [0,1]$ by

$$(4) \qquad b_\theta(p) = \frac{\theta p}{1 - p + \theta p}\,.$$

[1]Exponential utility satisfies Assumption 1. Power utility does not satisfy Assumption 1, because both partial derivatives are infinite at the reference point and thus the Köbberling-Wakker index of loss aversion is not well defined. However, we will treat power utility separately in Appendix A. It will be seen that, for this case, even stronger results may be obtained than those based solely on Assumption 1.

FIGURE 1.— Assumption 2 and the Benchmark Weighting Functions



This figure plots the benchmark functions for gains ($b_\lambda(p)$, dashed line) and losses ($b_{1/\lambda}(p)$, dotted line) for $\lambda = 2.25$ with the Tversky-Kahneman weighting function $w(p) = \frac{p^\delta}{(p^\delta+(1-p)^\delta)^{1/\delta}}$ (solid line) for $\delta = 0.65$. The light gray dashed line indicates the 45-degree line. w intersects with $b_\lambda(p)$ ($b_{1/\lambda}(p)$) at $p \approx 7.2\%$ ($p \approx 80,0\%$). Therefore, if $w^+ = w^- \equiv w$, then Assumption 2 is satisfied because—for example—$w^+(5\%) > b_\lambda(5\%)$ and $w^-(95\%) < b_{1/\lambda}(95\%)$.

We make the following assumption on the weighting function:

ASSUMPTION 2 (Weighting Functions) *There exists at least one* $p \in (0, 1)$ *such that*

1. $w^+(p) > \frac{\lambda p}{1-p+\lambda p} \equiv b_\lambda(p)$ *and*
2. $w^-(1-p) < \frac{1-p}{1-p+\lambda p} \equiv b_{1/\lambda}(1-p)$.[2]

The functions b_θ are strictly increasing with $b_\theta(0) = 0$ and $b_\theta(1) = 1$, and therefore are weighting functions themselves. We refer to b_λ and $b_{1/\lambda}(p)$ as the *benchmark weighting function* for gains and losses, respectively. b_θ intersects with the 45-degree line at zero and one. For $\theta = 1$, b_θ coincides with the 45-degree line. For $\theta > 1$ ($\theta < 1$), b_θ is strictly concave (convex) and lies above (below) the 45-degree line. Figure 1 plots the gain and loss benchmark weighting functions to illustrate Assumption 2. Assumption 2 requires that at least one probability p is "sufficiently" overweighted by w^+ in the sense that $w^+(p) > b_\lambda(p) \geq p$ and the complementary probability $1-p$ is "sufficiently" underweighted by w^- in the sense that $w^-(1-p) < b_{1/\lambda}(1-p) \leq 1-p$.

[2]Note that $b_{1/\lambda}(1-p) = 1 - b_\lambda(p)$ so that condition 2 could also be written as $1 - w^-(1-p) > b_\lambda(p)$.

By considering the tangent lines of the gain (loss) benchmark weighting function at zero (at one) one obtains that Assumption 2 follows from the simpler but stronger condition that there exists a $p \in (0,1)$ such that

$$w^+(p) > \lambda p \text{ and } w^-(1-p) < \lambda(1-p) + (1-\lambda).$$

As is apparent from this latter condition, a yet stronger sufficient condition for Assumption 2 is

$$\limsup_{p \to 0} \frac{w^+(p)}{p} > \lambda \text{ and } \limsup_{p \to 0} \frac{1 - w^-(1-p)}{p} > \lambda,$$

i.e., the gain (loss) weighting function is steeper than the gain (loss) benchmark weighting function at zero (one). These *limit superior conditions* ensure that Assumption 2 is met for all p close to zero. Intuitively, an arbitrarily small probability p is overweighted by more than the index of loss aversion λ. If w^+ and w^- are differentiable at zero and one, then the limit superior conditions simplify to

$$w^{+\prime}(0) > \lambda \text{ and } w^{-\prime}(1) > \lambda.$$

If these derivatives do not exist because they approach infinity (or because w^+ has a jump at zero and w^- has a jump at one), the limit superior is infinite. Also in this case, Assumption 2 is met.

Assumption 2, in particular, is satisfied by the commonly used, inverse-S-shaped weighting functions proposed by Tversky and Kahneman (1992, see Figure 1), Goldstein and Einhorn (1987), Prelec (1998), and the neo-additive weighting function (Wakker 2010, p. 208), for *all* parameter values. As these inverse-S-shaped weighting functions satisfy Assumption 2 for all sufficiently small probabilities, in our interpretations we take the view that Assumption 2 is met for small probabilities.[3]

Table I shows for the weighting functions mentioned the non-zero probabilities p for which conditions 1 and 2 of Assumption 2 are satisfied. While small, we note that these probabilities need not be extremely small. The maximal values displayed in the table are easily obtained as the intersection of $w^+(p)$ with the benchmark weighting function $b_\lambda(p)$, or as the intersection of $w^-(1-p)$ with $b_{1/\lambda}(1-p)$, whichever is smaller. For the example depicted in Figure 1, $w^+(p)$ intersects with $b(p)$ at approximately 7.2%, which yields the upper left entry of Table I.

In the following, our only assumptions on the CPT preference functional (2) are Assumptions 1 and 2. The restrictions imposed on the value function are very mild and of technical nature. The results of this paper may thus be seen as consequences of probability weighting

[3]It is interesting to note, however, that also S-shaped weighting functions, as are sometimes observed in empirical settings for some decision makers, may well satisfy Assumption 2. These weighting functions would satisfy Assumption 2 for large probabilities.

TABLE I

PROBABILITIES p FOR WHICH ASSUMPTION 2 IS SATISFIED

	Tversky-Kahneman	Prelec	Goldstein-Einhorn
$\lambda = 2.25$	$p \leq 7.2\%$	$p \leq 6.2\%$	$p \leq 3.5\%$
$\lambda = 1.5$	$p \leq 17.6\%$	$p \leq 13.9\%$	$p \leq 10.4\%$

Notes. Table I shows strictly positive probabilities p for which Assumption 2 is satisfied for pronounced loss aversion ($\lambda = 2.25$) and milder loss aversion ($\lambda = 1.5$). Each column is for a different parametric choice of the weighting function, which is taken to be the same for gains and losses ($w^+ \equiv w^-$). The first (second, third) column shows this interval for the Tversky-Kahneman weighting function with parameter $\delta = 0.65$ (Prelec with parameters $b = 1.05$ and $a = 0.65$, Goldstein-Einhorn with parameters $a = 0.69$ and $b = 0.77$). For the functional forms as well as for a motivation of the parameter choices, see Wakker (2012, pp. 206-208).

alone. Prospect theory without any probability weighting is excluded from the subsequent discussion. Introducing probability weighting as little as specified in Assumption 2, however, is sufficient for the results below—irrespective of the particular shape of the value and weighting function.

3. STATIC RESULTS

3.1. *Prospect Theory's Skewness Preference in the Small*

This paper starts out with a seemingly innocuous result on prospect theory preferences and small, skewed risks. We say that a risk is *attractive* or that an agent *wants to take a risk* if the CPT utility of current wealth plus the risk is *strictly* higher than the CPT utility of current wealth.

THEOREM 1 (Prospect Theory's Skewness Preference in the Small) *For every wealth level* x, *every* $\epsilon > 0$, *there exists an attractive zero-mean binary lottery* $L \equiv L(p, b, a)$ *with* $a, b \in (-\epsilon, +\epsilon)$, *i.e.,* L *may be arbitrarily small.*

The proof in Appendix B explicitly constructs such a lottery. The probability parameter p of this lottery is the probability p that exists according to Assumption 2. By a means of a continuity argument we obtain the following corollary.

COROLLARY 1 (Unfair Attractive Gambles) *For every wealth level* $x \in \mathbb{R}$ *there exists an attractive, arbitrarily small binary lottery with negative mean.*

It is straightforward to formulate a local version of Theorem 1.

COROLLARY 2 (Local Result) *At some given wealth level* x *there exists an attractive, arbitrarily small zero-mean binary lottery even if Assumption 2 is relaxed by replacing* $\lambda =$

$\sup_{x\in\mathbb{R}} \frac{\partial_- U(x)}{\partial_+ U(x)}$ *with* $\frac{\partial_- U(x)}{\partial_+ U(x)}$. *If* U *is differentiable at* x, *then Assumption 2 may be further relaxed to: There exists at least one* $p \in (0,1)$ *such that* $w^+(p) > p$ *and* $w^-(1-p) < 1-p$.

Theorem 1 relates to skewness preference. A binary lottery is right-skewed if and only if the high payoff occurs with the smaller probability. If the prospect theory weighting functions are inverse-S-shaped, Assumption 2 is met for sufficiently small p. Therefore, we can interpret the proof of Theorem 1 as the construction of a sufficiently right-skewed, fair binary lottery.

Skewness preference has been of major interest in the recent economics and finance literature. Numerous empirical and experimental papers find support for skewness preference, e.g., Kraus and Litzenberger (1976) and Boyer et al. (2010) for asset returns, Golec and Tamarkin (1998) for horse-race bets, and Ebert and Wiesen (2011) in a laboratory experiment. In many of these situations, prospect theory may do a good job in explaining behavior because it implies skewness preference. Other papers have argued like this. To best of our knowledge, however, Theorem 1 is the first rigorous result that relates CPT to skewness preference. Before closing this section, we show that skewness preference is so strong that it prevents an unambiguous statement on risk attitudes within CPT.

3.2. *Prospect Theory Agents are Risk-Averse and Risk-Seeking Nowhere*

A theory-free definition of risk aversion (risk-seeking) at a given wealth level is that any zero-mean risk is unattractive (attractive) to the agent. Theorem 1 thus implies:

Corollary 3 *At any wealth level a CPT agent is not risk-averse.*

As in the Corollary 2, Assumptions 1 and 2 may be relaxed to obtain a tighter result locally, i.e., at a specific wealth level x. It is also straightforward to formulate analogous versions of Theorem 1 and its corollaries on the unattractiveness of left-skewed gambles and risk-seeking under CPT. The precise assumptions[4] are complementary to our Assumptions 1 and 2, and are likewise fulfilled by the common choices for the value and weighting functions mentioned before. Then, at any wealth level, there exists an arbitrarily small, left-skewed binary risk which is unattractive so that a CPT agent is not risk-seeking.

All together, at any wealth level CPT agents are neither risk-seeking nor risk averse, but are risk seeking for some risks and risk averse for others. To best of our knowledge, a formal proof of this result has not been presented before. Also note that it is not obvious, for example, that probability weighting is sufficient for risk-taking at wealth levels where the value function is very concave. In particular, probability weighting as mild as given by

[4] $\tilde{\lambda} = \inf_{x\in\mathbb{R}} \partial_- U(x)/\partial_+ U(x) > 0$ must be well-defined, i.e., the decision maker must not be "infinitely gain seeking." Moreover, for some p, $w^+(p) < b_{\tilde{\lambda}}(p)$ and $w^-(1-p) > b_{1/\tilde{\lambda}}(1-p)$. Evidently, these properties are also necessary for the famous inverse-S-shape, and fulfilled for the weighting functions mentioned earlier for probabilities p close to one. Therefore, the unattractive risk constructed with the same methodology as in the proof of Theorem 1 is left-skewed.

Assumption 2 is sufficient. Assuming inverse-S-shaped weighting functions, the conditions in Assumption 2 are met for small probabilities (and the conditions in footnote 4 are met for large probabilities). Assuming in addition an S-shaped value function, our results imply the fourfold pattern of prospect theory as hypothesized by Tversky and Kahneman (1992, abstract): "risk aversion for gains and risk seeking for losses of high probability; risk seeking for gains and risk aversion for losses of low probability."

In Appendix A we show that results similar to those presented in this section also apply to the special case of a power-S-shaped value function as proposed by Tversky and Kahneman (1992), even though it does not satisfy Assumption 1. Moreover, in Appendix A we show that we may also have skewness preference in the *large*. The maximal size of an attractive gamble depends on further assumptions of the value function that we do not impose in this paper. However, a particularly striking result on preference for *large* attractive risks worth mentioning is obtained for the S-shaped power function. This result follows from a recent result due to Azevedo and Gottlieb (2012) when applied to skewness preference in the small. If the decision maker's reference point equals current wealth, then there exists an attractive, zero-mean binary risks of arbitrary size; see Appendix A for details.

4. ON PROSPECT THEORY IN A DYNAMIC CONTEXT

In this section, we investigate the consequences of skewness preference in the small in a dynamic context. Assume that Assumptions 1 and 2 are fulfilled. Alternatively, one may assume the power utility case (Assumptions 3 and 4 in Appendix A). We now define a stochastic process $(X_t)_{t\in\mathbb{R}_+}$ that could reflect the accumulated returns of an investment project, or the price development of an asset traded in the stock market. It could likewise model an agent's wealth when gambling in a casino. Let $(W_t)_{t\in\mathbb{R}_+}$ be a Brownian motion and $(X_t)_{t\in\mathbb{R}_+}$ a Markov diffusion that satisfies

$$\mathrm{d}X_t = \mu(X_t)\mathrm{d}t + \sigma(X_t)\mathrm{d}W_t$$

where we assume that $\mu : \mathbb{R} \to \mathbb{R}$ and $\sigma : \mathbb{R} \to (0, \infty)$ are such that there exists a unique solution with continuous paths.[5] Note that the most frequently considered processes, arithmetic and geometric Brownian motion, are covered by this definition.[6]

Investment or gambling strategies are modeled as stopping times, which are plans when to sell an asset or when to stop gambling. $\mathcal{S}$ denotes the set of all stopping times such that the agent bases his stopping decision only on his past observations from the gambling or investment experience. Formally, all $\tau \in \mathcal{S}$ are adapted to the natural filtration $(\mathcal{F}_t)_{t\in\mathbb{R}_+}$ of

[5] $\mu : \mathbb{R} \to \mathbb{R}$ and $\sigma : \mathbb{R} \to (0, \infty)$ are locally Lipschitz continuous Borel functions with linear growth, i.e., there exists a $K > 0$ such that $|\mu(x)|^2 + |\sigma(x)|^2 \leq K(1 + |x|^2)$.

[6] In Appendix D we discuss discounting. In particular, for the geometric Brownian motion we show explicitly that also the discounted stochastic process satisfies our assumptions.

the process $(X_t)_{t\in\mathbb{R}_+}$. The prospect theory utility of a stopping strategy $\tau \in \mathcal{S}$ given the information $\mathcal{F}_t$ at time t is given by the prospect theory value of the random variable X_τ that is induced by the stopping strategy τ :

$$CPT(X_\tau, \mathcal{F}_t) = \int_{\mathbb{R}_+} w^+(\mathbb{P}(u(X_\tau - r) > y \,|\, \mathcal{F}_t))\mathrm{d}y \\ - \int_{\mathbb{R}_-} w^-(\mathbb{P}(u(X_\tau - r) < y \,|\, \mathcal{F}_t))\mathrm{d}y\,.$$

The probability weighting of prospect theory induces a time inconsistency (Machina 1989). Therefore, an initially attractive investment strategy τ may later on be regarded as unattractive and thus be dismissed for another strategy. A *naïve* investor is time-inconsistent and also unaware of this time-inconsistency. Therefore, he does not anticipate that later on he might deviate from his initial investment plan. Formally, at every point in time, the naïve CPT agent looks for a gambling or investment strategy τ that brings him higher CPT utility than stopping immediately. If such a strategy exists, he continues the investment—irrespective of his earlier plan. In the following, we always consider such a naïve agent.

The naïve agent stops at time t if and only if his prospect value $CPT(X_\tau, \mathcal{F}_t)$ of any stopping strategy $\tau \in \mathcal{S}$ is less than or equal to $CPT(X_t, \mathcal{F}_t) = U(X_t)$, which is what he gets from stopping immediately. Formally, the naïve CPT agent stops at t, if and only if

$$U(X_t) \geq \sup_{\tau \geq t} CPT(X_\tau, \mathcal{F}_t).$$

Note that not stopping at time zero means that the agent finds gambling or investing attractive in the first place.

The following theorem presents a general and extreme result on the gambling or investment behavior of a naïve CPT agent with a continuous, infinite time horizon. In Section 5 we show that it has far-reaching implications by applying it to three selected dynamic decision problems. In Section 6 we explain why the result typically also applies in discrete and finite time. Keep in mind that the result is independent of the particular CPT specification, with only mild assumptions on the value and weighting functions. No assumption is made regarding the agent's reference point, and this reference point may change over time.[7] The result holds for the general class of stochastic processes specified above, and also for processes with a very negative drift. Moreover, the result is global in the sense that it applies at any time t, for whatever current wealth X_t, irrespectively of the gambling or investment history.

[7] To avoid confusion, also the dynamic results in this and the next section hold for a "classical" prospect theory reference point. However, this reference point may change over time and evolve according to any $\mathcal{F}_t-$adapted stochastic reference point process $(r_t)_{t\in\mathbb{R}_+}$. This implies that the reference point is constant and known to the agent when the stopping decision has to be made. As such, the reference point may depend on the past investment evolution and past behavior, but not on future behavior as is the case for the stochastic, expectations-based reference points considered Kőszegi and Rabin (2006, 2007).

Theorem 2 (Main Result) *The naïve agent never stops.*

The intuition of the proof is to construct a stop-loss strategy, i.e., one where the agent plans to stop if the process falls a little bit and plans to continue until it has risen significantly. This results in a right-skewed binary risk which the agent prefers to stopping immediately due to Theorem 1 or, in the case of a power value function, due to Theorem 3 in Appendix A.

5. APPLICATIONS

5.1. *Casino Gambling*

Our first example is the continuous, infinite time horizon analogue to the discrete, finite time casino gambling model of Barberis (2012). Barberis studies the behavior of prospect theory agents who gamble 50-50 bets in a casino for up to five periods. Because no analytical solution is available, the author investigates planned and actual behavior by computing the CPT values of all possible gambling strategies that can be generated by a five-period, 50-50 binomial tree, for more than 8000 parameter combinations of the CPT parametrization of Tversky and Kahneman (1992). The reference point is constant and assumed to equal initial wealth when entering the casino. An advantage of this approach is that it also yields results on the behavior of sophisticated agents with and without commitment. For naïvité, in this setting, the simulation results show that gamblers typically plan to follow a stop-loss strategy when entering the casino, but end up playing a gain-exit strategy (i.e., continue gambling when losing and stop gambling when winning). We now first give the analytical solution to the continuous, infinite time analogue of the casino gambling model (which applies to the general version of CPT and the general class of stochastic processes defined before). We will compare with Barberis (2012) in Section 6 when we discuss finite and discrete time.

Let $(X_t)_{t\in\mathbb{R}_+}$ be an arithmetic Brownian motion with negative drift $\mu(x) = \mu < 0$ and constant variance $\sigma(x) = \sigma > 0$, i.e.,

$$\mathrm{d}X_t = \mu \mathrm{d}t + \sigma \mathrm{d}W_t \,.$$

Due to the negative drift the agent loses money in expectation if he does not stop. Further assume that the process absorbs at zero since then the agent goes bankrupt. From Theorem 2 it follows that the naïve agent gambles until the bitter end, i.e., he will continue gambling unless forced to due to bankruptcy. From standard results in probability theory we know that this will happen almost surely.

5.2. *Exercising an American Option*

Let $(X_t)_{t\in\mathbb{R}_+}$ be a geometric Brownian motion with drift $\mu < 0$ and variance $\sigma > 0$, i.e.,

$$\mathrm{d}X_t = X_t(\mu \mathrm{d}t + \sigma \mathrm{d}W_t)$$

The agent holds an American option that pays $X_t - K$ if exercised at time t. Here $K \in \mathbb{R}_+$ represents the costs of investment. The American option could be interpreted as an investment opportunity, i.e., a real option (compare Dixit and Pindyck 1994).The agent is allowed to exercise his option at every point in time $t \geq 0$. If the agent does not exercise the option, he receives a payoff of zero.

From Theorem 2 it follows that the agent will never exercise his option and hence the naïve prospect theory agent gets a payoff of zero even though he could get a strictly positive payoff by exercising the option immediately whenever $X_0 > K$.

5.3. *Prospect Theory Predicts no Disposition Effect for naïve Investors*

The disposition effect (Shefrin and Statman 1985) refers to individual investors being more inclined to sell stocks that have gained in value (winners) rather than stocks that have declined in value (losers). Numerous papers have addressed this phenomenon, and some of the most immediate explanations such as transaction costs, tax concerns, or portfolio rebalancing have been formidably ruled out by Odean (1998).

Several papers have investigated whether prospect theory can explain the disposition effect. However, all of them seem to have done so without the consideration of probability weighting (e.g. Kyle et al. 2006, Henderson 2012). Barberis (2012) notes that the binomial tree in his paper, which models a casino, may likewise represent the evolution of a stock price over time. Then, naïve investors may exhibit a disposition effect, even though they plan to do the opposite of the disposition effect. Our result can be related to the disposition effect in the same spirit.

We have proven that, under just some probability weighting, naïve CPT agents will sell neither losers nor winners at any time. As a consequence, prospect theory with probability distortion does not predict a disposition effect for naïve investors. This is especially striking as Henderson (2012) shows that, in an identical model without probability distortion, prospect theory can explain the disposition effect.

Note that continuous time price processes such as geometric Brownian motion that are covered by our setup fit particularly well for financial market models. In any case, in the next section we show that our result applies to a wide range of continuous or discrete, finite or infinite time horizon processes.

6. ROBUSTNESS TO DISCRETE AND FINITE TIME SPECIFICATIONS

While it may seem that our results hinge upon the continuous time setup, they do not. Continuous time ensures that at every point in time the strategy set of the agent is sufficiently rich. To illustrate this point consider a binomial random walk $(X_t)_{t\in\mathbb{N}}$ with jump size one and equal probability for up- and down movements. At every point in time t the agent can choose the stakesize $s_t \in [0,1]$ (as a fraction of his wealth y_t) to bet. The evolution of his wealth is then given by

$$y_{t+1} = y_t + s_t y_t (X_{t+1} - X_t)$$

with initial wealth $y_0 > 0$. The following strategy (of choosing s_t) results in any given fair binary lottery $L(p,b,a)$. Choose s_t maximal such that $y_{t+1} \in [a,b]$, i.e.,

$$s_t = \max\{\tilde{s} \in [0,1] : (1-\tilde{s})y_t \geq a \text{ and } (1+\tilde{s})y_t \leq b\} = \min\{1 - \frac{a}{y_t}, \frac{b}{y_t} - 1\}\,.$$

Due to the martingale property it follows from Doob's optional sampling theorem that the probabilities of hitting b and a that are induced by this strategy are fair, i.e., are p and $1-p$, respectively. Therefore, if $L(p,b,a)$ is attractive according to either Theorem 1 or 3, then the agent will gamble. Since an attractive lottery exists at any wealth level (i.e., at any time t) the agent never stops.

The crucial point of this example is that the *time space* may be discrete as long as the set of gambling strategies is sufficiently rich. Specifically, a global result like Theorem 2 requires that, at any time t, for any state X_t, at least one stopping strategy is available that results in an attractive (skewed) gamble. This explains why the gambling behavior over 50-50 bets for up to five periods is different. In this case, there is no skewness in the first place: the basic one-shot gamble is 50-50. Moreover, the generation of skewness through gambling a stop-loss strategy takes time, and thus is only feasible in the beginning. This *combination* of symmetric gambles and finite (very short) time horizon ensures that the "casino dries out of skewness." At some exogenous point in time, the casino does not allow for gambling strategies any more that result in attractive gambles.

With this in mind it is immediate that typically we also have never stopping for a finite time space. To this means, the casino (or the financial market) must offer an attractive (sufficiently skewed) gamble in a single period, i.e., in the final period. The commonly made assumption of complete markets (which says that securities with any payoff structure are available) yields our extreme prediction of never stopping for any time horizon.

In Appendix C we illustrate this point through a numerical example. There we assume the original finite, discrete time setting introduced by Barberis (2012), and only change the probability of an up-movement in the binomial tree from 1/2 to 1/37. In the casino paradigm, this corresponds to the assumption that a casino offers bets on a single number in French Roulette. We show that an agent with CPT preferences as in Tversky and Kahneman (1992) and the parameter estimates from that paper never stops gambling for any finite or infinite time horizon.

7. Conclusion

This paper derives results on prospect theory when probabilities are distorted at least a bit. The reference point is immaterial to our results and may change over time, and only a few natural assumptions are made on the value function. Moreover, the assumptions we impose on the probability weighting functions—while being the driving force of this paper—are also very mild. We propose a new weighting function which we call the benchmark weighting function. Our results apply to any version of CPT whose weighting function for gains lies above this benchmark weighting function for at least one probability; a similar condition is imposed on the weighting function for losses. All commonly used weighting functions satisfy these conditions.

We first prove that probability weighting implies skewness preference in the small. At any wealth level, a CPT agent wants to take a sufficiently right-skewed binary risk that is arbitrarily small, even if it has negative expectation. A corollary is that CPT agents are not risk-averse, even if, for example, the value function is concave everywhere. While we prove the existence of small attractive risks, under additional assumptions on the value function attractive risks may also be large. For the power value function of Tversky and Kahneman (1992), for example, we show that when the reference point is current wealth there exists an attractive risk of arbitrary size.

These static results have consequences for CPT in a dynamic context. When prospect theory with probability weighting is taken to dynamic setups, the decision maker is inherently time-inconsistent, and the question arises of how this time-inconsistency can be resolved. We investigate the predictions of prospect theory for a naïve agent who is unaware of this time-inconsistency. For a very large class of gambles, naïve CPT agents will never stop gambling. The implications of this result are far-reaching and very extreme. Naïve agents gamble in a casino until the bitter end, i.e., they will go bankrupt almost surely. They will never exercise an American option, even if it is profitable to do so right from the beginning. And CPT does not predict the disposition effect for naïve agents. These results are formulated for a continuous, infinite time horizon, but generally extend to discrete and finite time.

A possible conclusion from this paper is that prospect theory with probability weighting in combination with naïvité leads to absurd predictions in many dynamic decision situations and is thus not a very promising candidate for explaining dynamic behavior.

APPENDIX A: THE CASE OF A S-SHAPED VALUE FUNCTION

In this section, we consider a power value function which satisfies the reflection property, equation (3).

Assumption 3 (S-Shaped Power Value Function) *The value function is given by*

$$U(x) = \begin{cases} (x-r)^\alpha, & \text{if } x \geq r \\ -\hat{\lambda}\left(-(x-r)\right)^\alpha, & \text{if } x < r \end{cases} \tag{5}$$

with $\alpha \in (0,1)$ *and* $\hat{\lambda} > 1$.[8]

For this very choice, the Köbberling-Wakker index of loss aversion $\frac{\partial_- U(r)}{\partial_+ U(r)}$ is not well-defined (in particular, it is not equal to $\hat{\lambda}$) because the power function has infinite derivative at 0. Therefore, Assumption 1 is not fulfilled, and thus Theorem 1 does not apply. However, we can state a similar result under a slightly modified assumption on the weighting functions.

Assumption 4 *There exists at least one* $p \in (0,1)$ *such that*

1. $w^+(p) > \frac{p^\alpha \hat{\lambda}}{(1-p)^\alpha + \hat{\lambda} p^\alpha}$ *and*
2. $w^-(1-p) < \frac{(1-p)^\alpha}{(1-p)^\alpha + \hat{\lambda} p^\alpha}$.

The functions $p \mapsto \hat{\lambda} p^\alpha / ((1-p)^\alpha + \hat{\lambda} p^\alpha)$ and $p \mapsto \hat{\lambda}^{-1} p^\alpha / ((1-p)^\alpha + \hat{\lambda}^{-1} p^\alpha)$ and thus serve as benchmark weighting functions for the particular case of an S-shaped power value function. These functions are, respectively, similar in shape to the benchmark weighting function $b_{\hat{\lambda}}(p)$ ($b_{1/\hat{\lambda}}(p)$), but lie above $b_{\hat{\lambda}}(p)$ (below $b_{1/\hat{\lambda}}(p)$) for $\alpha \in (0,1)$. Thus Assumption 4 is stronger than Assumption 2. Nevertheless, Assumption 4 is also met by the weighting functions of Tversky and Kahneman (1992) and Goldstein and Einhorn (1987) under parameter restrictions that are fulfilled according to most empirical studies; see Azevedo and Gottlieb (2012) for an elaboration.[9] For the weighting function of Prelec (1998), Assumption 4 is always true.

Table II shows, analogously to Table I, probabilities for which the conditions in Assumption 4 are fulfilled. We see that these probabilities are smaller than those in Table I, which evidences that Assumption 2 is tighter than Assumption 4. However, in all but one case these probabilities may still be larger than two percent.

Theorem 3 (Skewness Preference in the Small for the S-Shaped Power Value Function) *Assume Assumptions 3 and 4 instead of Assumptions 1 and 2. For every wealth level* x *and every* $\epsilon > 0$ *there exists an attractive, zero-mean binary lottery* $L \equiv L(p,b,a)$ *with* $a, b \in (-\epsilon, \epsilon)$, *i.e.,* L *may be arbitrarily small.*

Power utility is differentiable everywhere except at the reference point. Therefore, note that Corollary 2, which assumes just minimal probability weighting, also applies to power utility whenever we are not at the reference point. Therefore, we need Assumption 4 exclusively to cover gambling at the reference point.

We called Theorems 1 and 3 skewness preference in the *small*. However, attractive risks may also be large. Recall Assumptions 1 and 2 and note that these are global assumptions that ensure gambling at any wealth

[8]We do not consider the ill-posed specification of utility with different power parameters for gains and losses. In that case, $\hat{\lambda}$ does not capture loss aversion in a meaningful way as is formidably illustrated by Wakker (2010, pp. 267-270).

[9]Sufficient conditions for Assumption 4 are—similarly to the general case—$w^+(p) > p^\alpha \hat{\lambda}$ and $w^+(1-p) > p^\alpha \hat{\lambda}$, and also $\limsup_{p\to 0} \frac{w^+(p)}{p^\alpha} > \hat{\lambda}$ and $\limsup_{p\to 0} \frac{1-w^-(1-p)}{p^\alpha} > \hat{\lambda}$. Azevedo and Gottlieb discuss a condition similar to these limit superior conditions.

TABLE II

Probabilities p for which Assumption 4 is satisfied

	Tversky-Kahneman	Prelec	Goldstein-Einhorn
$\lambda = 2.25$	$p \leq 2.4\%$	$p \leq 2.8\%$	$p \leq 0.35\%$
$\lambda = 1.5$	$p \leq 10.1\%$	$p \leq 7.6\%$	$p \leq 2.9\%$

Notes. Table II shows strictly positive probabilities p for which Assumption 4 is satisfied for pronounced loss aversion ($\lambda = 2.25$) and milder loss aversion ($\lambda = 1.5$) when $\alpha = 0.88$. Each column is for a different parametric choice of the weighting function, which is taken to be the same for gains and losses ($w^+ \equiv w^-$). The first (second, third) column is for the Tversky-Kahneman weighting function with parameter $\delta = 0.65$ (Prelec with parameters $b = 1.05$ and $a = 0.65$, Goldstein-Einhorn with parameters $a = 0.69$ and $b = 0.77$). For the functional forms as well as for a motivation of the parameter choices, see Wakker (2012, pp. 206-208).

level. In particular, they ensure gambling at wealth levels where utility is very concave, which supports risk aversion. In other words, Assumption 2 on probability weighting is a sufficient condition for gambling at any wealth level, and for *whatever* value function with loss aversion parameter λ. The maximal size and skewness of attractive risks obviously depends on the specific choice of the value function and its shape at current wealth.

To illustrate this point, let us mention a result on large *large* skewed attractive risks for the S-shaped power function. It is obtained as a combination of our result with a far-reaching observation made recently by Azevedo and Gottlieb (2012). The authors consider a prospect theory agent with S-shaped power value function and whose reference point equals current wealth. Azevedo and Gottlieb (2012) show that for any attractive zero-mean binary gamble L, the multiple cL ($c > 1$) is also attractive. We, on the other hand, showed that there always exists an attractive risk which is arbitrarily small. Therefore:

Corollary 4 (Skewness Preference in the Small and in the Large for the S-Shaped Power Value Function) *Assume Assumptions 3 and 4 instead of Assumptions 1 and 2. If the decision maker's reference point equals current wealth, then there exists an attractive, zero-mean binary lottery of arbitrary size.*

APPENDIX B: PROOFS

B.1. *Proof of Theorem 1*

We split the proof into three cases $x > r$, $x < r$, and $x = r$. We prove the equivalent result that for all $x \in \mathbb{R}$ and every $\epsilon > 0$ there exists a binary lottery $L = L(p, b, a)$ with mean x and $a, b \in (x - \epsilon, x + \epsilon)$ such that $CPT(L) > CPT(x)$. L having mean x yields

$$x = (1-p)a + pb \;\Leftrightarrow\; p = \frac{x-a}{b-a}\,.$$

Proof of case 1 ($x > r$). Choose $a > r$ such that both a and b are gains. Then lottery L gives the agent a utility of $CPT(L) = w^+(p)U(b) + (1 - w^+(p))U(a)$. Therefore, the agent prefers L over x if there exist $a < x$

and $b > x$ such that

$$0 < \left(1 - w^+\left(\frac{x-a}{b-a}\right)\right)U(a) + w^+\left(\frac{x-a}{b-a}\right)U(b) - U(x)$$
$$= (U(b) - U(a))\left(w^+\left(\frac{x-a}{b-a}\right) - \frac{U(x) - U(a)}{U(b) - U(a)}\right)$$
$$= \underbrace{p\,(U(b) - U(a))}_{>0}\left(\frac{w^+(p)}{p} - \frac{\frac{U(x)-U(a)}{x-a}}{\frac{U(b)-U(a)}{b-a}}\right). \tag{6}$$

Consider sequences $(a_n, b_n)_{n\in\mathbb{N}}$ with $a_n = x - \frac{p}{n}$ and $b_n = x + \frac{1-p}{n}$. Note that by construction

$$\frac{U(b_n) - U(a_n)}{b_n - a_n} = \frac{U(b_n) - U(x)}{b_n - x}\frac{b_n - x}{b_n - a_n} + \frac{U(x) - U(a_n)}{x - a_n}\frac{x - a_n}{b_n - a_n}$$
$$= \frac{U(b_n) - U(x)}{b_n - x}(1 - p) + \frac{U(x) - U(a_n)}{x - a_n}p.$$

Therefore, according to equation (6), the agent prefers lottery $L(p, b_n, a_n)$ over x if

$$0 < \frac{w^+(p)}{p} - \frac{\frac{U(x)-U(a_n)}{x-a_n}}{\frac{U(b_n)-U(x)}{b_n-x}(1-p) + \frac{U(x)-U(a_n)}{x-a_n}p}.$$

Note that

$$\lim_{n\to\infty} \frac{\frac{U(x)-U(a_n)}{x-a_n}}{\frac{U(b_n)-U(x)}{b_n-x}(1-p) + \frac{U(x)-U(a_n)}{x-a_n}p} = \frac{\partial_- U(x)}{\partial_+ U(x)(1-p) + \partial_- U(x)p} = \frac{\frac{\partial_- U(x)}{\partial_+ U(x)}}{1 - p + \frac{\partial_- U(x)}{\partial_+ U(x)}p} \le \frac{\lambda}{1 - p + p\lambda}.$$

Therefore, for n sufficiently large the agent finds lottery $L(p, b_n, a_n)$ attractive if

$$0 < \frac{w^+(p)}{p} - \frac{\lambda}{1 - p + p\lambda} \iff w^+(p) > \frac{\lambda p}{1 - p + p\lambda}, \tag{7}$$

and condition 1 of Assumption 2 ensures that there exists at least one such p.

Proof of case 2 ($x < r$). Choose $b < r$ such that both a and b are losses. In that case, lottery $L = L(p, b, a)$ secures the agent a utility of

$$CPT(L) = (1 - w^-(1 - p))U(b) + w^-(1 - p)U(a)$$

with $1 - p = \frac{b-x}{b-a}$. Therefore, the agent continues gambling if there exist $a < x$ and $b > x$ such that

$$0 < \left(1 - w^-\left(\frac{b-x}{b-a}\right)\right)U(b) + w^-\left(\frac{b-x}{b-a}\right)U(a) - U(x)$$
$$= U(b) - U(a) + U(a) - U(x) - w^-\left(\frac{b-x}{b-a}\right)(U(b) - U(a))$$
$$= (U(b) - U(a))\left(1 - w^-\left(\frac{b-x}{b-a}\right) + \frac{U(a) - U(x)}{U(b) - U(a)}\right)$$
$$= \underbrace{p\,(U(b) - U(a))}_{>0}\left(\frac{1 - w^-(1-p)}{p} - \frac{\frac{U(x)-U(a)}{x-a}}{\frac{U(b)-U(a)}{b-a}}\right) \tag{8}$$

which is the analogue to equation (6). The proof continues similar to that of case 1.

Proof of case 3 ($x = r$). When $x = r$, a is a loss and b is a gain. Therefore, L secures the agent a utility of

$$CPT(L) = w^-(1-p)U(a) + w^+(p)U(b)\,.$$

Note that, since $x = r$ by definition $U(x) = U(r) = 0$. Therefore, the agent chooses L over x if there exist $a < x$ and $b > x$ such that

$$\begin{aligned} 0 &< w^-(1-p)U(a) + w^+(p)U(b) - U(x) \\ &= w^+(p)\left(U(b) - U(a)\right) + \left(U(a) - U(x)\right)\left(w^-(1-p) + w^+(p)\right) \\ &= \left(U(b) - U(a)\right)\left(w^+(p) - \frac{U(x) - U(a)}{U(b) - U(a)}\left(w^-(1-p) + w^+(p)\right)\right) \\ &= \underbrace{p\left(U(b) - U(a)\right)}_{>0}\left(\frac{w^+(p)}{p} - \frac{\frac{U(x)-U(a)}{x-a}}{\frac{U(b)-U(a)}{b-a}}\left(w^-(1-p) + w^+(p)\right)\right). \end{aligned} \tag{9}$$

Similarly to before it can be shown that the agent prefers lottery $L(p, b_n, a_n)$ over x for large enough n if

$$0 < \frac{w^+(p)}{p} - \frac{\lambda}{1 - p + p\lambda}\left(w^-(1-p) + w^+(p)\right), \tag{10}$$

which is the analogue to what equation (7) is for case 1 of the proof. We conclude the proof by verifying

$$w^+(p) > \lambda p \frac{w^-(1-p)}{1-p}, \tag{11}$$

which is equivalent to Equation (10). The first (second) inequality below follows from condition 1 (condition 2) of Assumption 2.

$$w^+(p) > \frac{\lambda p}{1 - p + p\lambda} = \frac{\lambda p}{1-p} \cdot \frac{1-p}{1 - p + \lambda p} > \frac{\lambda p}{1-p} w^-(1-p),$$

which is equation (11). □

B.2. *Proof of Corollary 1*

The claim follows from continuity of the CPT preference functional (Assumption 1). □

B.3. *Proof of Corollary 2*

The first statement is evident from the proof of Theorem 1. The second statement follows because $\frac{\partial_- U(x)}{\partial_+ U(x)} = 1$ if U is differentiable at x, and because $b_\lambda(p) = b_{1/\lambda}(p) = p$ for $\lambda = 1$. □

B.4. *Proof of Corollary 3*

The statement is a direct consequence of Corollary 2: At any wealth level, there exists an attractive zero-mean risk. □

B.5. *Proof of Theorem 2*

Suppose the agent arrives at wealth x at time t, i.e., $X_t = x$. The agent can stop and get a utility of $CPT(x)$, or she may continue gambling. She continues gambling if there exists a gambling strategy $\tau \in \mathcal{S}$, i.e., a stopping time such that $CPT(x) < CPT(X_\tau)$. We consider strategies $\tau_{a,b}$ with two absorbing endpoints $a < x < b$ which stop if the process $(X_t)_{t\in\mathbb{R}_+}$ leaves the interval (a, b), i.e.,

$$\tau_{a,b} = \inf\{s \geq t : X_s \notin (a, b)\} .$$

Denote with $p = \mathbb{P}(X_{\tau_{a,b}} = b)$ the probability that with strategy $\tau_{a,b}$ the agent will stop at b. Note that strategy $\tau_{a,b}$ results in a binary lottery for the agent. We first prove that the agent never stops if $(X_t)_{t\in\mathbb{R}_+}$ is a martingale. For every stopping time $\tau_{a,b}$ consider the sequence of bounded stopping times $\min\{\tau_{a,b}, n\}$ for $n \in \mathbb{N}$. By Doobs optional stopping theorem (Revuz and Yor 1999, p. 70), $\mathbb{E}(X_{\min\{\tau_{a,b},n\}}) = X_t = x$. By the theorem of dominated convergence it follows that

$$\mathbb{E}\left(X_{\tau_{a,b}}\right) = \mathbb{E}\left(\lim_{n\to\infty} X_{\min\{\tau_{a,b},n\}}\right) = \lim_{n\to\infty} \mathbb{E}\left(X_{\min\{\tau_{a,b},n\}}\right) = x .$$

Hence, $X_{\tau_{a,b}}$ implements the binary lottery $L(p, a, b)$ with expectation x. From Theorem 1 (Theorem 3) it follows that there exist $a, b \in \mathcal{I}$ such that the agent prefers the binary lottery $L(p, a, b)$ induced by the strategy $\tau_{a,b}$ over the certain outcome x.

In the last step we prove that the naïve agent never stops even if $(X_t)_{t\in\mathbb{R}_+}$ is not a martingale. Define the strictly increasing scale function $S : \mathbb{R} \to \mathbb{R}$ by

$$S(x) = \int_0^x \exp\left(-\int_0^y \frac{2\mu(z)}{\sigma^2(z)} \mathrm{d}z\right) \mathrm{d}y .$$

Define a new process $\hat{X}_t = S(X_t)$ and a new value function $\hat{U}(x) = (U \circ S^{-1})(x)$. Note that the loss aversion index of the value function $\hat{U}$ equals the loss aversion index of U because

$$\frac{\partial_-\hat{U}(x)}{\partial_+\hat{U}(x)} = \frac{\partial_-\hat{U}(x)S'(x)}{\partial_+\hat{U}(x)S'(x)} = \frac{\partial_-U(x)}{\partial_+U(x)} .$$

A CPT agent with the value function $\hat{U}$ facing the process $(\hat{X}_t)_{t\in\mathbb{R}_+}$ evaluates all stopping times exactly as a CPT agent with value function U who faces $(X_t)_{t\in\mathbb{R}_+}$. The process $\hat{X}_t = S(X_t)$ satisfies (Revuz and Yor 1999, p. 303 ff)

$$\mathbb{P}\left(\hat{X}_{\tau_{a,b}} = S(b)\right) = \mathbb{P}\left(X_{\tau_{a,b}} = b\right) = \frac{S(x) - S(a)}{S(b) - S(a)} ,$$

and hence it follows from the argument for martingales that the agent never stops. □

B.6. *Proof of Theorem 3*

Since U is differentiable everywhere except at r, the result for $x \neq r$ follows from Corollary 2. Thus suppose $x = r$ and let $a_n = x - p/n$ and $b_n = x + (1 - p)/n$ (as in the proof of Theorem 1). For the power-S-shaped value function it is easily seen that

$$\frac{U(x) - U(a_n)}{x - a_n} = \frac{0 + \hat{\lambda}\left(\frac{p}{n}\right)^\alpha}{\frac{p}{n}} = \hat{\lambda} n^{1-\alpha} p^{\alpha-1}$$

and

$$\frac{U(b_n) - U(a_n)}{b_n - a_n} = \frac{\left(\frac{1-p}{n}\right)^\alpha + \hat{\lambda}\left(\frac{p}{n}\right)^\alpha}{\frac{1-p}{n} + \frac{p}{n}} = n^{1-\alpha}\left((1 - p)^\alpha + \hat{\lambda} p^\alpha\right)$$

such that

$$\frac{\frac{U(x)-U(a)}{x-a}}{\frac{U(b)-U(a)}{b-a}} = \hat{\lambda} \frac{p^{\alpha-1}}{\left((1-p)^\alpha + \hat{\lambda} p^\alpha\right)}.$$

Therefore, according to equation (9), $L(p, b_n, a_n)$ is attractive for large enough n if

$$0 < \left(\frac{w^+(p)}{p^\alpha} - \hat{\lambda} \frac{1}{\left((1-p)^\alpha + \hat{\lambda} p^\alpha\right)} \left(w^-(1-p) + w^+(p)\right) \right) \Longleftrightarrow w^+(p) > \hat{\lambda} p^\alpha \frac{w^-(1-p)}{(1-p)^\alpha}.$$

The proof concludes analogously to that of case 3 of Theorem 1 by employing Assumption 4. □

B.7. *Proof of Corollary 4*

According to Theorem 3 there exists an attractive, arbitrarily small binary risk. According to Azevedo and Gottlieb (2012, p. 1294) it can be scaled up to any size. □

APPENDIX C: EXAMPLE FOR NEVER STOPPING IN DISCRETE AND FINITE TIME

Consider the five-period binomial decision tree of Barberis (2012). Assume a casino that offers a fair version of French Roulette. We assume a fair casino to be close to the model of Barberis (2012). Then the basic gamble considered by Barberis is the fair analogue to a bet on Red or Black, which occur with equal probability. Now suppose the agent can also bet on a single number, which occurs with probability $\frac{1}{37}$. Consider an agent who only considers to bet 10 units of money on a single number. He is not even able to form a gambling strategy over several periods. This implies a rather coarse strategy space, a feature which is actually working against our "never stopping" result. However, the basic gamble is skewed whereas the basic gamble in Barberis (2012) is symmetric. Let $(X_t)_{t\in\mathbb{R}_+}$ be the binomial random walk that represents his wealth. It increases by 360 with probability $\frac{1}{37}$ and decreases by 10 with probability $\frac{36}{37}$, starting at some level $X_0 \in \mathbb{R}$, i.e.,

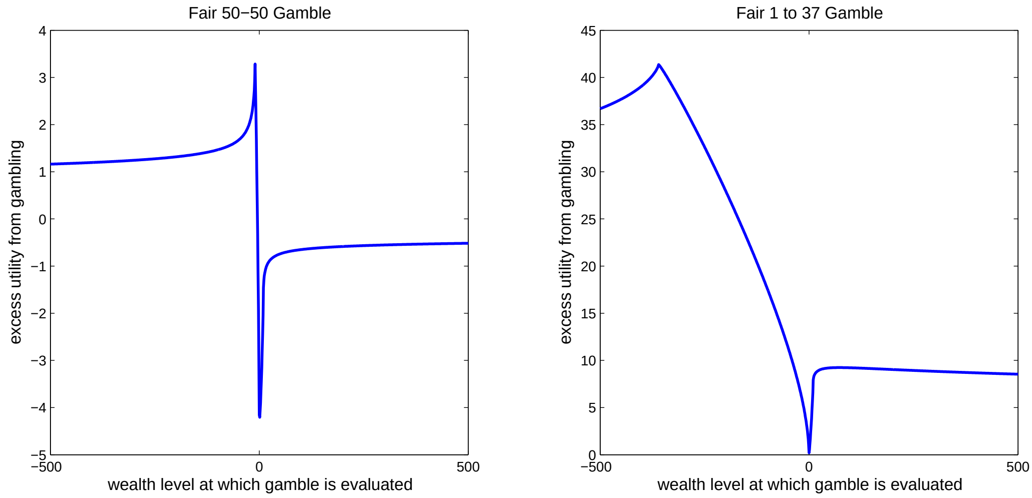
$$\mathbb{P}(X_{t+1} = X_t + 360) = \frac{1}{37} \text{ and } \mathbb{P}(X_{t+1} = X_t - 10) = \frac{36}{37}.$$

The agent is forced to stop in the final period T, which is exogenous, or if the random walk reaches zero.

Suppose the agent has CPT preferences given by the original parametrization of Tversky and Kahneman (1992) with parameters as estimated by the authors. Figure C plots the excess utility from gambling for the two basic gambles described above, as a function of current wealth. For the 50-50 gamble (left panel), gambling is attractive over the area of losses, and unattractive at the reference point and thereafter. This fits with the common intuition of risk seeking over losses and risk aversion over gains, which is induced by the S-shaped value function.

Note that the probability weighting component has no grip when evaluating 50-50 gambles. However, the right panel shows that gambling the skewed basic gamble is attractive *everywhere*. The lowest utility from gambling is at the reference point, but this utility is still positive (the exact value is +0.56). Therefore, at any node of the binomial tree, the agent will want to gamble. That is, the agent never stops even though we have finite time with an *arbitrary* number of gambling periods and a rather limited strategy space. Only one basic gamble is available, but this gamble is sufficiently skewed to be attractive to this very CPT agent. A stop-loss plan would grant even higher utility to the agent, but the one-shot gamble is attractive in itself already.

FIGURE 2.— Gambling utility for a symmetric and a skewed gamble



This figure shows the excess utility an agent gains from gambling (over not gambling) for different wealth levels. The left panel shows the utility from gambling a fair 50-50 bet, while the right panel shows the utility from gambling a fair 1 to 37 bet. The agent is a CPT maximizer with the parametrization of Tversky and Kahneman (1992) with parameters given by $\alpha = 0.88$, $\delta = 0.65$, and $\lambda = 2.25$. The agent's reference point is 0.

APPENDIX D: DISCOUNTING FOR GEOMETRIC BROWNIAN MOTION

In this section, we investigate the robustness of our results to discounting and provide formal results for the important case of geometric Brownian motion. Geometric Brownian motion is often used to model the price development of an asset. Thus it is interesting to consider the discounted process $(Z_t)_{t\in\mathbb{R}_+} = (e^{-\alpha t}X_t)_{t\in\mathbb{R}_+}$ where α denotes the opportunity costs of investment, for example the risk-free rate. Recall that the geometric Brownian motion $(X_t)_{t\in\mathbb{R}_+}$ with drift μ and volatility σ solves

$$\mathrm{d}X_t = X_t\left(\mu \mathrm{d}t + \sigma \mathrm{d}W_t\right) .$$

For the geometric Brownian motion it is a well known result that $(Z_t)_{t\in\mathbb{R}_+}$ solves $\mathrm{d}Z_t = Z_t\left([\mu - r]\mathrm{d}t + \sigma \mathrm{d}W_t\right)$, which is again a Markov diffusion (a geometric Brownian motion with a different drift). Thus also the discounted process $(Z_t)_{t\in\mathbb{R}_+}$ is covered by our setup and the "never stopping" result, Theorem 2, applies. For processes $(X_t)_{t\in\mathbb{R}_+}$ other than geometric Brownian motion it may also be true that $(Z_t)_{t\in\mathbb{R}_+} = (e^{-\alpha t}X_t)_{t\in\mathbb{R}_+}$ is a Markov diffusion, but proofs may be less tractable. Another approach to incorporate discounting would be to generalize Theorem 2 to more general stochastic processes, which may likewise be possible under some technical conditions.

REFERENCES

AZEVEDO, E. AND D. GOTTLIEB (2012): "Risk-Neutral Firms Can Extract Unbounded Profits from Consumers with Prospect Theory Preferences," *Journal of Economic Theory*, 147, 1291–1299.

BARBERIS, N. (2012): "A Model of Casino Gambling," *Management Science*, 58, 35–51.

BARBERIS, N. AND W. XIONG (2009): "What Drives the Disposition Effect? An Analysis of a Long-Standing Preference-Based Explanation," *Journal of Finance*, 64, 751–784.

BERNOULLI, D. (1738/1954): "Specimen Theoriae Novae de Mensura Sortis. Commentarii Academiae Scientiarum Imperialis Petropolitanae. Translated, Bernoulli, D., 1954. Exposition of a New Theory on the Measurement of Risk." *Econometrica*, 22, 23–36.

Boyer, B., T. Mitton, and K. Vorkink (2010): "Expected Idiosyncratic Skewness," *Review of Financial Studies*, 23, 169–202.

de Giorgi, E. and T. Hens (2006): "Making Prospect Theory Fit for Finance," *Financial Markets and Portfolio Management*, 20, 339–360.

Dixit, A. K. and R. S. Pindyck (1994): *Investment under Uncertainty*, vol. 15, Princeton University Press.

Ebert, S. and D. Wiesen (2011): "Testing for Prudence and Skewness Seeking," *Management Science*, 57, 1334–1349.

Goldstein, W. and H. Einhorn (1987): "Expression Theory and the Preference Reversal Phenomena," *Psychological Review*, 94, 236–254.

Golec, J. and M. Tamarkin (1998): "Bettors Love Skewness, Not Risk, at the Horse Track," *Journal of Political Economy*, 106, 205–225.

Gottlieb, D. (2013): "Prospect Theory, Life Insurance, and Annuities," Working Paper, University of Pennsylvania.

Henderson, V. (2012): "Prospect Theory, Liquidation and the Disposition Effect," *Management Science*, 58, 445–460.

Kahneman, D. and A. Tversky (1979): "Prospect Theory: An Analysis of Decision under Risk," *Econometrica*, 47, 263–292.

Kőszegi, B. and M. Rabin (2006): "A Model of Reference-Dependent Preferences," *Quarterly Journal of Economics*, 121, 1133–1166.

——— (2007): "Reference-Dependent Risk Attitudes," *American Economic Review*, 97, 1047–1073.

Köbberling, V. and P. Wakker (2005): "An Index of Loss Aversion," *Journal of Economic Theory*, 122, 119–131.

Kraus, A. and R. Litzenberger (1976): "Skewness Preference and the Valuation of Risky Assets," *Journal of Finance*, 31, 1085–1100.

Kyle, A., H. Ou-Yang, and W. Xiong (2006): "Prospect Theory and Liquidation Decisions," *Journal of Economic Theory*, 129, 273–288.

Machina, M. (1989): "Dynamic Consistency and Non-Expected Utility Models of Choice under Uncertainty," *Jorunal of Economic Literature*, 27, 1622–1668.

Odean, T. (1998): "Are Investors Reluctant to Realize Their Losses?" *Journal of Finance*, 53, 1775–1799.

Prelec, D. (1998): "The Probability Weighting Function," *Econometrica*, 66, 497–527.

Revuz, D. and M. Yor (1999): *Continuous Martingales and Brownian Motion*, Springer.

Shefrin, H. and M. Statman (1985): "The Disposition to Sell Winners Too Early and Ride Losers Too Long: Theory and Evidence," *Journal of Finance*, 40, 777–790.

Tversky, A. and D. Kahneman (1992): "Advances in Prospect Theory: Cumulative Representation of Uncertainty," *Journal of Risk and Uncertainty*, 5, 297–323.

von Neumann, J. and O. Morgenstern (1944): *Theory of Games and Economic Behavior*, Princeton University Press.

Wakker, P. (2010): *Prospect Theory for Risk and Ambiguity*, Cambridge University Press.

Xu, Z. and X. Zhou (2013): "Optimal Stopping under Probability Distortion," *Annals of Applied Probability*, 23, 251–282.