Asset Pricing with

Second-Order Esscher Transforms

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☐ **The purpose of the paper is**:

- to introduce in the general **Econometric Asset Pricing Modelling (EAPM)** setting, formalized by Bertholon, Monfort and Pegoraro (2008), and based on the exponential-affine SDF modelling principle,

$i)$ **a wider bridge between the historical ($\mathbb{P}$) and the risk-neutral ($\mathbb{Q}$)** dynamics of **state vector** (w_t) (say), and

$ii)$ to preserve, at the same time, its **tractability and flexibility**.

☐ **This goal is achieved by introducing** the notion of

- **Exponential-Quadratic stochastic discount factor (SDF)**, or equivalently of **Second-Order Esscher Transform**.

- **exploiting** one the EAPM strategies (the **Back Modelling**, in particular).

☐ **The log-pricing kernel** is specified

- as a **quadratic function of the factor** (w_t), and the associated sources of risk

- are priced by means of (possibly) **non-linear stochastic first-order and second-order (say) risk-correction coefficients** (when the **Back Modelling** strategy is adopted).

☐ Focusing on security market models

- this approach is developed in two important multivariate stochastic frameworks:

- **the conditionally Gaussian framework**, and **the conditionally Gaussian Switching Regimes setting**.

☐ **With respect to the classical** exponential-affine SDF approach,

- our specification **makes possible to price** not only mean-based and regime-shift risks but also **variance-covariance-based sources of risk**, and this is obtained regardless **the presence of homoscedasticity or conditional heteroscedasticity** in the factor's dynamics.

☐ **Compared with the continuous-time approach**, our methodology :

- provides a (diffusion-like) **risk-neutral conditional variance-covariance matrix** of the factor **different from the historical one**

- **keeping** the the historical and risk-neutral probabilities equivalent

- while, **in continuous time, they would be mutually singular**.

☐ Moreover, **our asset pricing setting shows** (contrary to QG and WAR asset pricing models):

- regardless the **functional specification of the risk-correction coefficients**,

- a historical factor dynamics **still conditionally Gaussian or Regime-Switching Gaussian** and, thus,

- **computationally tractable** (i.e. a likelihood function known exactly in closed form or by standard filtering techniques).

☐ This approach could be used not only in option pricing models, but also for instance in **interest rate** and **credit risk models**.

Outline of the Talk :

0. **Compound Autoregressive Processes**
1. **The Second-Order Esscher Transform of a probability density function**
2. **The Exponential-Quadratic SDF modelling principle**
3. **Conditionally Gaussian Economies**
4. **Conditionally Gaussian Switching Regimes Economies**

0.1 Compound Autoregressive of order 1 (Car(1)) Processes

□ *A n-dimensional process $\{y_t\}$ is called Car(1) if its conditional Laplace transform $\varphi_t(u \,|\, \underline{y_t}) = E[\exp(u'y_{t+1}) \,|\, \underline{y_t}]$ is of the form:*

$$\varphi_t(u \,|\, \underline{y_t}) = E_t[\exp(u'y_{t+1})] = \exp[a_t(u)'y_t + b_t(u)]\,, \quad u \in \mathbb{R}^n\,, \tag{1}$$

where a_t and b_t may depend on t in a deterministic way.

- The Log-Laplace transform $\psi_t(u \,|\, \underline{y_t}) = \text{Log}\,\varphi_t(u \,|\, \underline{y_t})$ is therefore affine in y_t, which implies that all the conditional cumulants.

- In particular the conditional mean and the conditional variance-covariance matrix, are affine in y_t.

Gaussian AR(1) and VAR(1) processes

☐ If y_{t+1} is a Gaussian AR(1) process defined by:

$$y_{t+1} = \mu + \rho y_t + \varepsilon_{t+1}$$

where ε_{t+1} is a gaussian white noise distributed as $\mathcal{N}(0, \sigma^2)$, then the process is Car(1) with $a(u) = u\rho$ and $b(u) = u\mu + \frac{\sigma^2}{2}u^2$.

If y_{t+1} is a Gaussian VAR(1) process defined by:

$$y_{t+1} = \mu + \Phi y_t + \varepsilon_{t+1}$$

where ε_{t+1} is a gaussian white noise distributed as $\mathcal{N}(0, \Omega)$, then the process is Car(1) with $a(u) = \Phi' u$ and $b(u) = u'\mu + \frac{1}{2}u' \, \Omega \, u$.

Homogeneous Markov Chains

☐ Let us consider a J-state homogeneous Markov Chain z_{t+1}, which can take the values $e_j \in \mathbb{R}^J$, $j \in \{1, \ldots, J\}$;

→ e_j is the j^{th} column of the $(J \times J)$ identity matrix I_J;

→ the transition probability, from state e_i to state e_j is

$$\pi_{i,j} = \pi(e_i, e_j) = Pr(z_{t+1} = e_j \,|\, z_t = e_i)$$

- The process $\{z_t\}$ is a Car(1) process with:

$$\begin{aligned} a_z(u, \pi) &= \left[\log\left(\sum_{j=1}^{J} \exp(u' e_j) \pi(e_1, e_j)\right), \ldots, \log\left(\sum_{j=1}^{J} \exp(u' e_j) \pi(e_J, e_j)\right)\right]', \\ b(u) &= 0\,. \end{aligned}$$

0.2 Univariate Switching Regimes Car(1) processes

- The procedure we follow for the construction of scalar Switching Regimes Car(1) processes is the following.

- First, let us consider a J-states homogeneous Markov Chain z_{t+1}, which can take the values $e_j \in \mathbb{R}^J$, $j \in \{1, \ldots, J\}$, where e_j is the j^{th} column of the $(J \times J)$ identity matrix. The transition probability, from state e_i to state e_j, is $\pi(e_i, e_j) = Pr(z_{t+1} = e_j \,|\, z_t = e_i)$.

- We have seen that z_{t+1} is a Car(1) process.

- Second, let us consider a univariate Car(1) process with a conditional Laplace transform given by $\exp\left[a(u)\, x_t + b(u)\right]$, and let us assume that $b(u)$ can be written:

$$
\begin{aligned}
b(u) &= \tilde{b}(u)'\lambda \quad \text{where} \\
\tilde{b}(u) &= (b_1(u), \ldots, b_m(u))' \text{ and } \lambda = (\lambda_1, \ldots, \lambda_m)'\,.
\end{aligned}
$$

- Third, let us assume that the parameters λ_i are stochastic and linear functions of z_{t+1}:

$$
\begin{aligned}
&\lambda_i(z_{t+1}) = \lambda_i' z_{t+1} \ \forall\ i \in \{1, \ldots, m\}\,, \\
&\lambda_i \in \mathbb{R}^J\,.
\end{aligned}
$$

- Then the conditional distribution of x_{t+1}, given $\underline{x_t}$ and $\underline{z_{t+1}}$, has a Laplace transform given by:

$$E[\exp(ux_{t+1})|\, \underline{x_t}\,, \underline{z_{t+1}}] = \exp\left[a(u)\, x_t + \tilde{b}(u)'\Lambda z_{t+1}\right]\ ,\ \Lambda = \begin{bmatrix} \lambda_1' \\ \vdots \\ \lambda_m' \end{bmatrix} \text{ is a } (m, J)\text{-matrix.} \tag{2}$$

- *The conditional Laplace transform of* $(x_{t+1}, z_{t+1}')'$ *given* $\underline{x_t}\,, \underline{z_t}$ *has the following form*:

$$\varphi_t(u, v) \;=\; E\left[\exp(ux_{t+1} + v'z_{t+1})\,|\, \underline{z_t},\, \underline{x_t}\right] = \exp\left\{a(u)\, x_t + a_z(v + \Lambda'\tilde{b}(u), \pi)'z_t\right\}\ ,$$

and thus $(x_{t+1}, z_{t+1}')'$ *is a Car(1) process.*

- **Proof:**

$$\varphi_t(u,v) = E[\exp(ux_{t+1} + v'z_{t+1}) \,|\, \underline{x_t}\,, \underline{z_t}]$$

$$= E\left\{E[\exp(ux_{t+1} + v'z_{t+1}) \,|\, \underline{x_t}\,, \underline{z_{t+1}}] \,|\, \underline{x_t}\,, \underline{z_t}\right\}$$

$$= E\left\{\exp(v'z_{t+1})\, E[\exp(ux_{t+1}) \,|\, \underline{x_t}\,, \underline{z_{t+1}}] \,|\, \underline{x_t}\,, \underline{z_t}\right\}$$

$$= E[\exp(v'z_{t+1} + a(u)\,x_t + \tilde{b}(u)'\Lambda z_{t+1}) \,|\, \underline{x_t}\,, \underline{z_t}]$$

$$= \exp[a(u)\,x_t]\, E\{\exp[(v + \Lambda'\tilde{b}(u))'z_{t+1}] \,|\, \underline{x_t}\,, \underline{z_t}]$$

$$= \exp\left[a(u)\,x_t + a_z(v + \Lambda'\tilde{b}(u), \pi)'z_t\right]\,.$$

Gaussian AR(1) process

- $a(u) = u\,\rho$, $\tilde{b}(u)' = \left(u, \dfrac{u^2}{2}\right)$ and $\lambda' = (\nu, \sigma^2)$. Thus, $m = 2$.

- $\Lambda = \begin{bmatrix} \nu' \\ \sigma^{2\prime} \end{bmatrix}$ is a $(2, J)$-matrix, where $\nu = (\nu_1, \ldots, \nu_J)'$ and $\sigma^2 = (\sigma_1^2, \ldots, \sigma_J^2)'$.

- The dynamics of x_{t+1}, conditionally to $\underline{x_t}\,, \underline{z_{t+1}}$, is given by:

$$x_{t+1} = \nu' z_{t+1} + \rho\, x_t + (\sigma^{2\prime} z_{t+1})^{1/2} \varepsilon_{t+1}\,,$$

where ε_{t+1} is a gaussian white noise distributed as $\mathcal{N}(0, 1)$.

Gaussian AR(1) process

- The joint conditional Laplace transform is:

$$
\begin{aligned}
\varphi_t(u,v) &= \exp\left[a(u)\,x_t + a_z(v + \Lambda'\tilde{b}(u), \pi)' z_t\right] \\
&= \exp\left[u\,\rho\,x_t + a_z\left(v + u\,\nu + \frac{u^2}{2}\sigma^2, \pi\right)' z_t\right]
\end{aligned}
$$

1. The Second-Order Esscher Transform of a probability density function

- Let us introduce **a new family of probability distributions**, associated with the p.d.f. f, having the (classical, First-Order) Esscher Transform as a subset.

- This new family, that we call **Second-Order Esscher Transforms**, is built upon the concept of **Second-Order Laplace Transform**.

- It gives the possibility, for instance, **TO MODIFY NOT ONLY** the mean **BUT ALSO the variance-covariance matrix** of a multivariate Gaussian distribution or the mean and the variance-covariance matrix of the components of a mixture of multivariate Gaussian distributions.

- **The Second-Order Laplace Transform of the p.d.f. $f(y)$ is**:

$$\varphi_S(\theta_1, \theta_2) = \int_{\mathbb{R}^n} f(y) \exp(\theta_1' y + y' \theta_2 y) d\nu(y)$$

$\theta_1 \in \mathbb{R}^n$, $\theta_2 \in \mathcal{S}_n(\mathbb{R})$ an $(n \times n)$ real symmetric matrix and $\theta = (\theta_1, \theta_2) \in \Theta$, $\Theta = \{(\theta_1, \theta_2) \in \mathbb{R}^n \times \mathcal{S}_n(\mathbb{R}) : \int_{\mathbb{R}^n} f(y) \exp(\theta_1' y + y' \theta_2 y) d\nu(y) < \infty\}$ being the definition set.

- **the assumption $\theta_2 \in \mathcal{S}_n(\mathbb{R})$ is not a restriction** since any square matrix A (say) is the sum of a symmetric matrix $(A + A')/2$ and of an antisymmetric matrix $(A - A')/2$, and since a quadratic form associated to an antisymmetric matrix is equal to zero.

- **The Second-Order Esscher Transform of $\mathbb{P}$ associated with** (θ_1, θ_2), denoted by $S_{(\theta_1,\theta_2)}(\mathbb{P})$, is given by the family of probability distributions defined by the p.d.f.:

$$g(y; \theta_1, \theta_2) = \frac{f(y)\,\exp(\theta_1' y + y'\theta_2 y)}{\varphi_S(\theta_1, \theta_2)}. \tag{3}$$

- If $\theta_2 = 0$, we have First-Order Esscher Transform: denoted by $F_{(\theta_1)}(\mathbb{P})$.

$i)$ **Multivariate Gaussian distribution :** $y \sim N(\mu, \Sigma)$

$$g(y; \theta_1, \theta_2) = \frac{1}{(2\pi)^{n/2}\sqrt{\det\left[(\Sigma^{-1} - 2\theta_2)^{-1}\right]}} \times$$

$$\exp\left[-\frac{1}{2}(y - (I - 2\Sigma\theta_2)^{-1}(\mu + \Sigma\theta_1))'(\Sigma^{-1} - 2\theta_2)(y - (I - 2\Sigma\theta_2)^{-1}(\mu + \Sigma\theta_1))\right],$$

▶ we find that $g(y;\theta_1,\theta_2)$ is the p.d.f. of the family of the n-dimensional Gaussian random variable

$$N[(I-2\Sigma\theta_2)^{-1}(\mu+\Sigma\theta_1),(\Sigma^{-1}-2\theta_2)^{-1}]$$

if $(\Sigma^{-1}-2\theta_2)$ is assumed to be a symmetric positive definite matrix.

- For any given (θ_1,θ_2), the Gaussian random variable generated by (3) has a different mean as well as a different variance-covariance matrix and any n-dimensional Gaussian distribution can be reached.

- If $\theta_2 = 0$, the new probability distribution has the same variance-covariance matrix as the starting one.

$ii)$ **Finite Mixtures of Multivariate Gaussian distributions :**

- given a finite mixture of n-dimensional Gaussian random variables with p.d.f. $f(y) = \sum_{j=1}^{J} \lambda_j \, n(y; \mu_j, \Sigma_j)$,

- the associated family of probability density functions generated by the Second-Order Esscher Transform is:

$$g(y; \theta_1, \theta_2) = \sum_{j=1}^{J} \lambda_j^* \, n(y; (I - 2\Sigma_j\theta_2)^{-1}(\mu_j + \Sigma_j\theta_1), (\Sigma_j^{-1} - 2\theta_2)^{-1}) \, ,$$

$$\text{with } \lambda_j^* = \frac{\lambda_j \varphi_{S,j}(\theta_1, \theta_2)}{\sum_{j=1}^{J} \lambda_j \varphi_{S,j}(\theta_1, \theta_2)} \, , \tag{4}$$

- where

$$
\begin{aligned}
&\varphi_{S,j}(\theta_1, \theta_2) \\
&= \int_{\mathbb{R}^n} \exp(\theta_1 y + y'\theta_2 y) n(y; \mu_j, \Sigma_j) dy \\
&= \exp\left[-\frac{1}{2}\log\det(I - 2\Sigma_j\theta_2) - \frac{1}{2}\mu_j'\Sigma_j^{-1}\mu_j + \frac{1}{2}(\Sigma_j^{-1}\mu_j + \theta_1)'(\Sigma_j^{-1} - 2\theta_2)^{-1}(\Sigma_j^{-1}\mu_j + \theta_1)\right],
\end{aligned}
$$

and $0 \leq \lambda_j^* \leq 1, \sum_{j=1}^{J} \lambda_j^* = 1$.

(5)

- This is the family of p.d.f. of a n-dimensional Finite Mixture of J Gaussian random variables $N((I-2\Sigma_j\theta_2)^{-1}(\mu_j+\Sigma_j\theta_1), (\Sigma_j^{-1}-2\theta_2)^{-1})$, $j \in \{1, \ldots, J\}$, having different mean and variance-covariance matrix w.r.t. the starting one (if $\theta_2 \neq 0$), as well as different mixing weights.

2. The Exponential-Quadratic Stochastic Discount Factor Modelling Principle

2.1 General Information and Historical Distribution

- In what follows, we consider an economy between dates 0 and T. The new information in the economy at date t is denoted by w_t, while $\underline{w_t} = (w_t, w_{t-1}, ..., w_0)$ is the entire information between 0 and t.

- The random vector w_t is called a factor or a state vector, its dimension is n and it can be made up of *latent or observable* variables (asset prices or macro variables).

- The historical dynamics of w_t is defined by the conditional distribution of w_{t+1} given $\underline{w}_t$, denoted by $\mathbb{P}_{t+1}$ (say) and characterized either

- by the p.d.f. $f_t(w_{t+1}|\underline{w}_t)$

- or the Laplace transform $\varphi_t(u|\underline{w}_t)$,

- or the Log-Laplace transform $\psi_t(u|\underline{w}_t) = \log[\varphi_t(u|\underline{w}_t)]$.

2.2 The Exponential-Affine Stochastic Discount Factor

- Assuming existence, linearity and continuity of the pricing function, and under the absence of arbitrage opportunity principle, there exists a (not unique in general) positive Stochastic Discount Factor (SDF) $M_{t,t+1}(\underline{w}_{t+1})$, for each $t \in \{0, \ldots, T-1\}$, such that the price at date t of the payoff $g(\underline{w}_s)$ delivered at $s > t$ is given by

$$p_t\left[g(\underline{w}_s)\right] = E_t\left[M_{t,t+1} \ldots M_{s-1,s}\, g(\underline{w}_s)\right]$$

[see Hansen and Richard (1987), Cochrane (2005) and Bertholon, Monfort and Pegoraro (2008)].

- The asset pricing literature has in general derived or specified $M_{t,t+1}(\underline{w}_{t+1})$ as an exponential-affine function of w_{t+1}:

$$M_{t,t+1} = \exp\left[-r_{t+1}(\underline{w}_t) + \alpha'_{1,t}(\underline{w}_t)w_{t+1} - \psi_t(\alpha_{1,t}|\underline{w}_t)\right]. \tag{6}$$

where $\alpha_{1,t}(\underline{w}_t)$ is the n-dimensional "risk-correction" or "risk-sensitivity" vector, also called the "market price" of factor risk.

- the Risk-Neutral (R.N.) conditional distribution of w_{t+1}, given $\underline{w}_t$ and denoted by $\mathbb{Q}_{t+1}$, has an exponential-affine (in w_{t+1}) p.d.f. with respect to $\mathbb{P}_{t+1}$ given by:

$$d_t^{\mathbb{Q}}(w_{t+1}|\underline{w}_t) = \frac{M_{t,t+1}(\underline{w}_{t+1})}{E_t\left[M_{t,t+1}(\underline{w}_{t+1})\right]} = \frac{\exp\left(\alpha'_{1,t}w_{t+1}\right)}{\varphi_t(\alpha_{1,t})} = \exp\left[\alpha'_{1,t}w_{t+1} - \psi_t(\alpha_{1,t})\right]. \tag{7}$$

- $f_t^{\mathbb{Q}}(w_{t+1}|\underline{w_t}) = f_t(w_{t+1}|\underline{w_t})\, d_t^{\mathbb{Q}}(w_{t+1}|\underline{w_t})$ and

  $\psi_t^{\mathbb{Q}}(u_1) = \psi_t(u_1 + \alpha_{1,t}) - \psi_t(\alpha_{1,t})$ $(u_1 \in \mathbb{R}^n)$

- Conversely, the p.d.f. of the conditional historical distribution with respect to the R.N. one is given by :

$$
\begin{aligned}
d_t^{\mathbb{P}}(w_{t+1}|\underline{w_t}) &= \frac{1}{d_t^{\mathbb{Q}}(w_{t+1}|\underline{w_t})} \\
&= \exp\left[-\alpha'_{1,t} w_{t+1} + \psi_t(\alpha_{1,t})\right] \\
&= \exp\left[-\alpha'_{1,t} w_{t+1} - \psi_t^{\mathbb{Q}}(-\alpha_{1,t})\right],
\end{aligned}
\tag{8}
$$

- since $\psi_t(u_1) = \psi_t^{\mathbb{Q}}(u_1 - \alpha_{1,t}) - \psi_t^{\mathbb{Q}}(-\alpha_{1,t})$.

2.3 The Exponential-Quadratic Stochastic Discount Factor

The General Case with Non-Linear Stochastic Risk-Correction Coefficients

- Let us introduce now the following exponential-quadratic SDF:

$$M_{t,t+1}^{(S)} = \exp\left[-r_{t+1}(\underline{w_t}) + \alpha_{1,t}'(\underline{w_t})w_{t+1} + w_{t+1}'\alpha_{2,t}(\underline{w_t})w_{t+1} - \psi_{S,t}(\alpha_{1,t}, \alpha_{2,t}|\underline{w_t})\right], \tag{9}$$

- with $\psi_{S,t}(\alpha_{1,t}, \alpha_{2,t}|\underline{w_t}) = \log \varphi_{S,t}(\alpha_{1,t}, \alpha_{2,t}|\underline{w_t})$,

  $\varphi_{S,t}(\alpha_{1,t}, \alpha_{2,t}|\underline{w_t}) = E_t[\exp(\alpha_{1,t}'w_{t+1} + w_{t+1}'\alpha_{2,t}w_{t+1})]$ the conditional second-order Log-Laplace transform

- and where $\alpha_{2,t}$ is a (time-varying) $(n \times n)$ symmetric matrix $(\alpha_{2,t} \in \mathcal{S}_n(\mathbb{R}))$.

- The Risk-Neutral (R.N.) conditional distribution $\mathbb{Q}_{t+1}$ of w_{t+1} given $\underline{w}_t$, has an exponential-quadratic (in w_{t+1}) p.d.f. with respect to $\mathbb{P}_{t+1}$ given by:

$$d_t^{\mathbb{Q},S}(w_{t+1}|\underline{w}_t) = \frac{M_{t,t+1}^{(S)}(\underline{w}_{t+1})}{E_t\left[M_{t,t+1}^{(S)}(\underline{w}_{t+1})\right]} = \exp\left[\alpha'_{1,t}w_{t+1} + w'_{t+1}\alpha_{2,t}w_{t+1} - \psi_{S,t}(\alpha_{1,t},\alpha_{2,t})\right], \tag{10}$$

- and, therefore, the R.N. conditional p.d.f. (with respect to the same measure as the corresponding conditional historical probability) is $f_t^{\mathbb{Q},S}(w_{t+1}|\underline{w}_t) = f_t(w_{t+1}|\underline{w}_t)d_t^{\mathbb{Q},S}(w_{t+1}|\underline{w}_t)$ and the R.N. conditional second-order Log-Laplace transform is:

$$\psi_{S,t}^{\mathbb{Q}}(u_1,u_2) = \psi_{S,t}(u_1+\alpha_{1,t}, u_2+\alpha_{2,t}) - \psi_{S,t}(\alpha_{1,t},\alpha_{2,t})\,,\ \ u_1 \in \mathbb{R}^n\,,\ \ u_2 \in \mathcal{S}_n(\mathbb{R})\,. \tag{11}$$

- Conversely, the p.d.f. of the conditional historical distribution with respect to the R.N. one is given by :

$$
\begin{aligned}
d_t^{\mathbb{P},S}(w_{t+1}|\underline{w_t}) &= \frac{1}{d_t^{\mathbb{Q},S}(w_{t+1}|\underline{w_t})} \\
&= \exp\left[-\alpha'_{1,t}w_{t+1} - w'_{t+1}\alpha_{2,t}w_{t+1} + \psi_{S,t}(\alpha_{1,t},\alpha_{2,t})\right] \qquad (12) \\
&= \exp\left[-\alpha'_{1,t}w_{t+1} - w'_{t+1}\alpha_{2,t}w_{t+1} - \psi^{\mathbb{Q}}_{S,t}(-\alpha_{1,t},-\alpha_{2,t})\right],
\end{aligned}
$$

- since $\psi_{S,t}(u_1,u_2) = \psi^{\mathbb{Q}}_{S,t}(u_1-\alpha_{1,t},u_2-\alpha_{2,t}) - \psi^{\mathbb{Q}}_{S,t}(-\alpha_{1,t},-\alpha_{2,t})$.

☐ We get the following **Proposition:**

- *If we consider the exponential-quadratic stochastic discount factor* $M_{t,t+1}^{(S)}$*, the risk-neutral conditional distribution* $\mathbb{Q}_{t+1}$ *of* w_{t+1}*, conditionally to* $\underline{w_t}$*, is the conditional Second-Order Esscher Transform of* $\mathbb{P}_{t+1}$ *associated with* $(\alpha_{1,t}, \alpha_{2,t})$*, that is* $\mathbb{Q}_{t+1} = S_{(\alpha_{1,t},\alpha_{2,t})}(\mathbb{P}_{t+1})$.

- *Conversely, the historical conditional distribution* $\mathbb{P}_{t+1}$ *is the conditional Second-Order Esscher Transform of* $\mathbb{Q}_{t+1}$ *associated with* $(-\alpha_{1,t}, -\alpha_{2,t})$*, that is* $\mathbb{P}_{t+1} = S_{(-\alpha_{1,t},-\alpha_{2,t})}(\mathbb{Q}_{t+1})$.

☐ We will see in the following sections that, if the Back Modelling strategy is adopted:

- $\alpha_{1,t}$ and $\alpha_{2,t}$, namely the *stochastic* risk-correction coefficients, are allowed to be *any non-linear* function of the *present and past* values of the factor w_t,

- while keeping the historical factor dynamics computationally tractable (i.e. providing a likelihood function in closed form or computable by standard filtering techniques).

☐ We will also see that our exponential-quadratic change of probability measure involves a risk-neutral conditional variance-covariance matrix of the factor *different from* the historical one while keeping at the same time the probability measure $\mathbb{Q}$ *equivalent to* $\mathbb{P}$.

- This kind of result can not be obtained by a continuous-time (Girsanov-based) approach, given that a risk-neutral diffusion term different from the historical one would imply $\mathbb{Q}$ and $\mathbb{P}$ mutually singular.

- See Steele (2000) and Cont and Tankov (2004).

Structural Justifications of $M_{t,t+1}^{(S)}$

a) Bakshi and Madan (2007), working in a simple static and univariate setting and aggregating the marginal rate of substitution of power utility investors that are either long or short the market index, determine an exponential-quadratic (in the scalar index return) economy-wide SDF when the risk-aversion parameter among agents ϕ (say) is normally distributed.

b) In a dynamic framework *à la* Epstein and Zin (1989), Hansen, Heaton and Li (2008) obtain an exponential-quadratic specification by linearizing the log-SDF around the log of the (explicit and exponential-affine) SDF computed when the inverse of the EIS parameter (namely, ρ) is assumed to be equal to one.

- This quadratic term can, thus, be associated to an economy able to provide a (statistically suggested) time-varying wealth-consumption ratio, while the exponential-affine SDF case ($\rho = 1$) obliges this ratio to be unrealistically constant.

- See Lettau and Ludvigson (2001) and Hansen, Heaton, Li and Roussanov (2007) for further details.

Internal Consistency Conditions

- The no-arbitrage discrete-time asset pricing setting based on an exponential-affine SDF $M_{t,t+1}$, conveniently provides explicit conditions, through the historical and R.N. Log-Laplace transforms ψ_t and $\psi_t^{\mathbb{Q}}$, to guarantee the internal consistency of the model [see BMP (2008) for details].

- These Internal Consistency Conditions (ICC) are easily extended to the case of an exponential-quadratic SDF $M^{(S)}_{t,t+1}(\underline{w}_{t+1})$.

- Let us consider, for instance, the situation in which the factor w_{t+1} contains (at least) a geometric stock return and in which the short rate r_{t+1} is exogenous.

- If $w_{j,t+1} = e_j' w_{t+1}$ is a scalar geometric return (e_j being the j^{th} column of the identity matrix $I_{n\times n}$) we must have:

$$\begin{aligned} \exp(-r_{t+1}) E_t^{\mathbb{Q}}[\exp(w_{j,t+1})] = 1 \quad & \iff \quad r_{t+1} = \psi_{S,t}^{\mathbb{Q}}(e_j, 0) \\ & \iff \quad r_{t+1} = \psi_{S,t}(\alpha_{1,t} + e_j, \alpha_{2,t}) - \psi_{S,t}(\alpha_{1,t}, \alpha_{2,t}) \,. \end{aligned} \tag{13}$$

3. Conditionally Gaussian Economies

3.1 Pricing *Mean-based* and *Variance-Covariance-based* Sources of Risk

- Let us assume that the factor w_t is a n-dimensional vector of geometric stock returns of risky assets, that is $w_{i,t+1} = \log(S_{i,t+1}/S_{i,t})$ for each $i \in \{1,\ldots,n\}$, where $S_{i,t}$ is the price at t of asset i. If we follow the Direct Modelling strategy formalized by Bertholon, Monfort and Pegoraro (2008), we first have to specify the historical dynamics ($\mathbb{P}_{t+1}$) of w_{t+1}. Assuming conditional normality, that is:

$$w_{t+1} | \underline{w}_t \overset{\mathbb{P}}{\sim} N(\mu_t, \Sigma_t), \tag{14}$$

we have to choose μ_t and Σ_t (VARMA models with GARCH-type noise).

- Second, we have to specify $\alpha_{1,t}$ and $\alpha_{2,t}$ appearing in the exponential-quadratic SDF (9) and to impose the ICC (13):

$$r_{t+1} = \psi_{S,t}(e_i + \alpha_{1,t}, \alpha_{2,t}) - \psi_{S,t}(\alpha_{1,t}, \alpha_{2,t})\,, \text{ where}$$

$$\psi_{S,t}(u_1, u_2) = -\frac{1}{2}\log\det\left(I - 2\Sigma_t u_2\right) - \frac{1}{2}\mu_t'\Sigma_t^{-1}\mu_t$$
$$+\frac{1}{2}(\Sigma_t^{-1}\mu_t + u_1)'(\Sigma_t^{-1} - 2u_2)^{-1}(\Sigma_t^{-1}\mu_t + u_1)\,,$$

- which implies :

$$\frac{1}{2}\,vdiag\left[(\Sigma_t^{-1} - 2\alpha_{2,t})^{-1}\right] + (I - 2\Sigma_t\alpha_{2,t})^{-1}(\mu_t + \Sigma_t\alpha_{1,t}) = r_{t+1}e\,, \tag{15}$$

where e denotes the n-dimensional unitary vector.

- The associated (R.N.) dynamics ($\mathbb{Q}_{t+1}$) is given by:

$$w_{t+1} | \underline{w}_t \overset{\mathbb{Q}}{\sim} N\left[(I - 2\Sigma_t \alpha_{2,t})^{-1}(\mu_t + \Sigma_t \alpha_{1,t}), (\Sigma_t^{-1} - 2\alpha_{2,t})^{-1}\right], \qquad (16)$$

that is, $\mathbb{Q}_{t+1} = S_{(\alpha_{1,t}, \alpha_{2,t})}(\mathbb{P}_{t+1})$.

- If we impose to (16) the ICC (15), we find that the R.N. dynamics compatible with no-arbitrage restrictions is:

$$N\left[r_{t+1}e - \frac{1}{2} vdiag\left((\Sigma_t^{-1} - 2\alpha_{2,t})^{-1}\right), (\Sigma_t^{-1} - 2\alpha_{2,t})^{-1}\right]. \qquad (17)$$

- It is important to stress that this exponential-quadratic SDF change of probability measure induces three relevant generalizations with respect to the exponential-affine one.

$i)$ it provides a different R.N. conditional mean and conditional variance-covariance matrix, namely:

$$\begin{aligned} \mu_t^{\mathbb{Q}} &= r_{t+1}e - \frac{1}{2} vdiag\,(\Sigma_t^{\mathbb{Q}}) \\ \Sigma_t^{\mathbb{Q}} &= (\Sigma_t^{-1} - 2\alpha_{2,t})^{-1}\,, \end{aligned} \tag{18}$$

because of the second-order risk-sensitivity function $\alpha_{2,t}$.

- On the contrary, in the continuous-time (Brownian motion-based) framework, the risk-neutral diffusion term has to be equal to the historical one ($\Sigma_t^{\mathbb{Q}} = \Sigma_t$, in our notation) in order to guarantee $\mathbb{Q}$ equivalent to $\mathbb{P}$, otherwise the two measures would be mutually singular.

ii) for a given historical dynamics (estimated using stock return observations) and under no-arbitrage restrictions, the Gaussian risk-neutral dynamics, contrary to the exponential-affine setting, still delivers free parameters (those specifying $\alpha_{2,t}$) adapted to match derivative prices.

iii) the time-varying risk-sensitivity vectors characterizing the SDF are given by :

$$\begin{aligned} \alpha_{2,t} &= \frac{1}{2}\Sigma_t^{-1}\left[\Sigma_t^{\mathbb{Q}} - \Sigma_t\right](\Sigma_t^{\mathbb{Q}})^{-1} = \frac{\Sigma_t^{-1} - (\Sigma_t^{\mathbb{Q}})^{-1}}{2}, \text{ and} \\ \alpha_{1,t} &= (\Sigma_t^{\mathbb{Q}})^{-1}\mu_t^{\mathbb{Q}} - \Sigma_t^{-1}\mu_t \ , \end{aligned} \tag{19}$$

and, therefore, can be seen respectively as a normalized measure of the historical and risk-neutral variance-covariance spread, and as a (variance-weighted) measure of the historical and risk-neutral mean spread.

3.2 Generalized Market Risk Premium (*GMRP*) and Second-Order Black and Scholes Pricing Formula

- In order to provide a more precise interpretation of the risk-sensitivity functions $\alpha_{1,t}$ and $\alpha_{2,t}$, let us define the scalar market risk premium between t and $t+1$, associated to any stock return $w_{i,t+1}$ for $i \in \{1, \ldots, n\}$, in the following way:

$$\lambda_{t,t+1}^{(i)} = \log E_t[\exp(w_{i,t+1})] - r_{t+1}\,,$$

- and let us collect them in the vector $\lambda_{t,t+1} = \left[\lambda_{t,t+1}^{(1)}, \ldots, \lambda_{t,t+1}^{(n)}\right]'$.

- Then, from (15), we can write the following n-dimensional generalized market risk premium (*GMRP*):

$$
\begin{aligned}
\lambda_{t,t+1} &= \mu_t + \tfrac{1}{2}\, vdiag\, \Sigma_t - r_{t+1}\, e \\
&= \mu_t - \mu_t^{\mathbb{Q}}(\alpha_{2,t}) - \tfrac{1}{2}\, vdiag\, (\Sigma_t^{\mathbb{Q}}(\alpha_{2,t}) - \Sigma_t) \\
&= \lambda_{t,t+1}^F + \left[\mu_t^{\mathbb{Q}}(0) - \mu_t^{\mathbb{Q}}(\alpha_{2,t})\right] - \tfrac{1}{2}\, vdiag\, (\Sigma_t^{\mathbb{Q}}(\alpha_{2,t}) - \Sigma_t)\,,
\end{aligned}
\tag{20}
$$

with $\lambda_{t,t+1}^F := (\mu_t - \mu_t^{\mathbb{Q}}(0)) = -\Sigma_t\, \alpha_{1,t}$ denoting the n-dimensional (first-order) risk premium associated to an exponential-affine SDF (and e denoting the n-dimensional unitary vector).

- Relation (20) shows the role played by $\alpha_{2,t}$, that is, the consequences on the risk premium played by the introduction of a quadratic term in the SDF:

$i)$ if we assume $\alpha_{2,t} = 0$ (an exponential-affine SDF) we find $\lambda_{t,t+1} = \lambda^F_{t,t+1}$, that is, the risk premium is (classically) determined comparing only historical and risk-neutral factor conditional means and $-\alpha_{1,t}$ can be interpreted as a first moment-based risk premium per unit of conditional variance-covariance;

$ii)$ if $\alpha_{2,t} \neq 0$, the size of $\lambda_{t,t+1}$ depends positively on the mean spread $(\mu_t - \mu_t^{\mathbb{Q}}(\alpha_{2,t}))$ and negatively on the variance-covariance spread $vdiag\,(\Sigma_t^{\mathbb{Q}}(\alpha_{2,t}) - \Sigma_t)$, that empirical evidence finds to be (both) positive [see, among the others, Bakshi and Madan (2006)].

- If we consider the particular scalar ($n = 1$) static case ($r_{t+1} = r, \sigma_t = \sigma, \alpha_{2,t} = \alpha_2$), we immediately find the following explicit *Second-Order Black and Scholes pricing formula* (for European Call options):

$$C_{BS}^{(S)}(t, h; K, S_t, r, \sigma^2, \alpha_2) = C_{BS}(t, h; K, S_t, r, (\sigma^{\mathbb{Q}})^2(\alpha_2)) \,, \tag{21}$$

- in which α_2 is an additional degree of freedom with respect to the classical Black and Scholes one ($\alpha_2 = 0$ implies $C_{BS}^{(S)}(t, h; K, S_t, r, \sigma^2, 0) = C_{BS}(t, h; K, S_t, r, \sigma^2)$).

- This source of flexibility can be further exploited by specifying $\alpha_{2,t}$ as a deterministic function of time, still leading to an explicit pricing formula. Moreover, we can easily propose, in a dynamic setting, richer Call option pricing formulas if we assume σ_t^2 and $\alpha_{2,t}$ functions of the date t information.

3.3 The Back Modelling Approach to Conditionally Gaussian Economies

- Let us now adopt the Back Modelling strategy opening the way for a *tractable and flexible* specification of the asset pricing model of interest. More precisely, let us assume that the R.N. dynamics ($\mathbb{Q}_{t+1}$) of w_{t+1} is given by:

$$w_{t+1} | \underline{w_t} \overset{\mathbb{Q}}{\sim} N\left(\mu_t^{\mathbb{Q}}, \Sigma_t^{\mathbb{Q}}\right), \tag{22}$$

- with the associated conditional second-order Log-Laplace transform

$$\begin{aligned} \psi_{S,t}^{\mathbb{Q}}(u_1, u_2) &= -\tfrac{1}{2} \log \det \left(I - 2\Sigma_t^{\mathbb{Q}} u_2\right) - \tfrac{1}{2} \mu_t^{\mathbb{Q}\prime} (\Sigma_t^{\mathbb{Q}})^{-1} \mu_t^{\mathbb{Q}} \\ &\quad + \tfrac{1}{2} [(\Sigma_t^{\mathbb{Q}})^{-1} \mu_t^{\mathbb{Q}} + u_1]' [(\Sigma_t^{\mathbb{Q}})^{-1} - 2u_2]^{-1} [(\Sigma_t^{\mathbb{Q}})^{-1} \mu_t^{\mathbb{Q}} + u_1] , \end{aligned} \tag{23}$$

- and we impose the ICC $\psi^{\mathbb{Q}}_{S,t}(e_i, 0) = r_{t+1}$ for all $i \in \{1, \ldots, n\}$, that is:

$$\begin{aligned} r_{t+1} &= -\tfrac{1}{2}\mu_t^{\mathbb{Q}\prime}(\Sigma_t^{\mathbb{Q}})^{-1}\mu_t^{\mathbb{Q}} + \tfrac{1}{2}[(\Sigma_t^{\mathbb{Q}})^{-1}\mu_t^{\mathbb{Q}} + e_i]'\Sigma_t^{\mathbb{Q}}[(\Sigma_t^{\mathbb{Q}})^{-1}\mu_t^{\mathbb{Q}} + e_i] \\ &= \tfrac{1}{2}e_i'\Sigma_t^{\mathbb{Q}}e_i + e_i'\mu_t^{\mathbb{Q}} \quad \forall\, i \in \{1, \ldots, n\}\,. \end{aligned} \tag{24}$$

- From (24) we have $\mu_t^{\mathbb{Q}} = r_{t+1}e - \frac{1}{2}vdiag\Sigma_t^{\mathbb{Q}}$ and, therefore, we find the (no-arbitrage) risk-neutral dynamics:

$$N\left[r_{t+1}e - \tfrac{1}{2}\,vdiag\,\Sigma_t^{\mathbb{Q}}, \Sigma_t^{\mathbb{Q}}\right], \tag{25}$$

- which is easily made affine (tractability) simply assuming, for instance, $\Sigma_t^{\mathbb{Q}} = \Sigma^{\mathbb{Q}}$ and without making any assumption or imposing any restriction on the risk-correction coefficients.

- Consequently, the associated historical dynamics $\mathbb{P}_{t+1}$ is given, for *any non-linear stochastic* risk-correction coefficients $\alpha_{1,t}$ and $\alpha_{2,t}$, by $\mathbb{P}_{t+1} = S_{(-\alpha_{1,t},-\alpha_{2,t})}(\mathbb{Q}_{t+1})$ and we have:

$$w_{t+1}|\,\underline{w_t} \overset{\mathbb{P}}{\sim} N\left(\mu_t, \Sigma_t\right)\ ,$$

$$\mu_t = (I + 2\Sigma_t^{\mathbb{Q}}\alpha_{2,t})^{-1}(r_{t+1}e - \tfrac{1}{2}\, vdiag\, \Sigma_t^{\mathbb{Q}} - \Sigma_t^{\mathbb{Q}}\alpha_{1,t})\,,\ \ \Sigma_t = ((\Sigma_t^{\mathbb{Q}})^{-1} + 2\alpha_{2,t})^{-1}\,. \tag{26}$$

- So, for any given R.N. conditionally Gaussian dynamics (possibly affine) and for any risk-correction coefficients specifications:

  i) historical dynamics is still conditionally Gaussian: computational tractability

  ii) any conditional mean and any conditional variance-covariance matrix can be reached thanks to the non-linearities of the risk-correction coefficients.

4. Conditionally Gaussian Switching Regimes Economies

- In this section we extend the results of the previous one to the case of a Conditionally Gaussian Switching Regime ($CGSR$, say) Economy.

- A security market model in which the dynamics of the relevant (quantitative) factor is described by a conditionally Gaussian regime-switching model

$\rightarrow$ with a conditional mean and conditional variance featuring a general dependence on contemporaneous and past factor values as well as on the regime-indicator function.

4.1 The Conditional Second-Order Esscher Transform of a $CGSR$ Model

- Let us consider the $(J+1)$-dimensional factor $w_{t+1} = (y_{t+1}, z'_{t+1})'$, where y_{t+1} is a scalar geometric return between t and $t+1$ and z_{t+1} is a J-state variable valued in $\mathcal{E} = \{e_1, \ldots, e_J\}$, where e_j is the j^{th} column of a $(J \times J)$ identity matrix (the generalization to a vector of returns is straightforward).

- We assume that the historical dynamics of w_{t+1} is described by the following generalized conditionally Gaussian switching regimes ($CGSR$, say) model:

$$
\begin{aligned}
& y_{t+1} = \mu_t(\underline{y_t}, z_t, z_{t+1}) + \sigma_t(\underline{y_t}, z_t, z_{t+1})\varepsilon_{t+1} \\
& \varepsilon_{t+1} | z_{t+1}, \underline{z_t}, \underline{y_t} \overset{\mathbb{P}}{\sim} N(0,1) \qquad (27) \\
& \mathbb{P}(z_{t+1} = e_j | z_t = e_i, \underline{z_{t-1}}, \underline{y_t}) = \pi_{i,j}(\underline{y_t}) = \pi_{i,j,t} \quad \forall (e_i, e_j) \in \mathcal{E} \times \mathcal{E} \, .
\end{aligned}
$$

- The historical distribution $\mathbb{P}_{i,t+1}$ (say) of $(y_{t+1}, z'_{t+1})'$, conditionally to $\underline{y_t}$ and $z_t = e_i$, has a p.d.f. given by:

$$f_t(y_{t+1}, e_j | \underline{y_t}, z_t = e_i) \quad = \quad n\left[y_{t+1}; \mu_t(\underline{y_t}, e_i, e_j), \sigma_t^2(\underline{y_t}, e_i, e_j)\right] \pi_{i,j,t} \,. \tag{28}$$

- The second-order Laplace transform of $w_{t+1} = (y_{t+1}, z'_{t+1})'$ is:

$$\varphi_{S,t}(u_1, u_2) \quad = \quad E\left[\exp\left(u'_1 z_{t+1} + u'_{2,1} z_{t+1} y_{t+1} + u_{2,2} y^2_{t+1}\right) \mid \underline{y_t}, \underline{z_t}\right] \,, \tag{29}$$

where $u_2 = (u'_{2,1}, u'_{2,2})'$.

- Using the notation $\mu_{i,j,t} = \mu_t(\underline{y_t}, e_i, e_j)$ and $\sigma_{i,j,t} = \sigma_t(\underline{y_t}, e_i, e_j)$, we obtain from (29):

$$\varphi_{S,t}(u_1,u_2) = \widetilde{\varphi}_{S,t}(u_1,u_2)' z_t \,, \text{ with}$$

$$\widetilde{\varphi}_{S,t}(u_1,u_2) = \left[\widetilde{\varphi}_{S,t,1}(u_1,u_2), \ldots, \widetilde{\varphi}_{S,t,J}(u_1,u_2)\right]' \,, \text{ and}$$

$$\begin{aligned}\widetilde{\varphi}_{S,t,i}(u_1,u_2) &= E\left[\exp\left(u_1' z_{t+1} + u_{2,1}' z_{t+1} y_{t+1} + u_{2,2} y_{t+1}^2\right) \mid z_t = e_i, \underline{z_{t-1}}, \underline{y_t}\right] \\ &= \sum_{j=1}^{J} \pi_{i,j}(\underline{y_t}) \exp(u_1' e_j) \widetilde{\varphi}_{S,t,i,j}\left(u_{2,1}' e_j, u_{2,2}\right) \,, \end{aligned} \tag{30}$$

and where :

$$\begin{aligned}\widetilde{\varphi}_{S,t,i,j}\left(u_{2,1}' e_j, u_{2,2}\right) &= E\left[\exp\left(u_{2,1}' e_j y_{t+1} + u_{2,2} y_{t+1}^2\right) \mid z_{t+1} = e_j, z_t = e_i, \underline{z_{t-1}}, \underline{y_t}\right] \\ &= \int_{\mathbb{R}} n(y_{t+1}; \mu_{i,j,t}, \sigma_{i,j,t}^2) \exp\left[(u_{2,1}' e_j) y_{t+1} + u_{2,2} y_{t+1}^2\right] dy_{t+1} \,. \end{aligned} \tag{31}$$

- The p.d.f. of the conditional Second-Order Esscher transform $S_{\theta_1,\theta_2}(\mathbb{P}_{i,t+1})$ is obtained,

- first, by multiplying the p.d.f. (28) by $\exp(\theta_1' z_{t+1} + \theta_{2,1}' z_{t+1} y_{t+1} + \theta_{2,2} y_{t+1}^2)$

- and then, this product is normalized by $\varphi_{S,t}(\theta_1, \theta_2)$.

- We have the following **Proposition:**

- *The p.d.f. of the family of probability distributions $\mathbb{P}^*_{i,t+1}$ (say) generated by the conditional Second-Order Esscher transform $S_{(\theta_1,\theta_2)}(\mathbb{P}_{i,t+1})$ applied to the p.d.f. (28) is given by:*

$$\begin{aligned} g_t(y_{t+1}, e_j | z_t = e_i, \underline{z_{t-1}}, \underline{y_t}) &= \frac{\pi_{i,j,t} \exp\left(\theta_1' e_j + \theta_{2,1}' e_j y_{t+1} + \theta_{2,2} y_{t+1}^2\right) n(y_{t+1}; \mu_{i,j,t}, \sigma^2_{i,j,t})}{\varphi_{S,t}(\theta_1, \theta_2)} \\ &= \pi^*_{i,j,t}\, n\left(y_{t+1}; \frac{\mu_{i,j,t} + \sigma^2_{i,j,t}\theta_{2,1}' e_j}{1 - 2\sigma^2_{i,j,t}\theta_{2,2}}, \frac{\sigma^2_{i,j,t}}{1 - 2\sigma^2_{i,j,t}\theta_{2,2}}\right), \end{aligned} \tag{32}$$

where

$$\pi^*_{i,j,t} = \pi^*_{i,j}(\underline{y_t}) = \frac{\pi_{i,j,t} \exp(\theta_1' e_j) \widetilde{\varphi}_{S,t,i,j}(\theta_{2,1}' e_j, \theta_{2,2})}{\sum_{j=1}^J \pi_{i,j,t} \exp(\theta_1' e_j) \widetilde{\varphi}_{S,t,i,j}(\theta_{2,1}' e_j, \theta_{2,2})},$$

with $\widetilde{\varphi}_{S,t,i,j}(u_1, u_2) = \exp\left[-\frac{1}{2}\log\left(1 - 2\sigma^2_{i,j,t} u_1\right) - \frac{1}{2}\frac{\mu^2_{i,j,t}}{\sigma^2_{i,j,t}} + \frac{1}{2}\frac{(\mu_{i,j,t} + \sigma^2_{i,j,t} u_1)^2}{(\sigma^2_{i,j,t} - 2\sigma^4_{i,j,t} u_2)}\right].$

- From this Proposition we see that the family of Second-Order Esscher transform distribution of $(y_{t+1}, z'_{t+1})'$, conditionally to $\underline{y_t}, \underline{z_t}$, is defined by:

$$y_{t+1} = \mu_t^*(\underline{y_t}, z_t, z_{t+1}) + \sigma_t^*(\underline{y_t}, z_t, z_{t+1})\xi_{t+1}\,, \tag{33}$$

- where $\mu_t^*(\underline{y_t}, e_i, e_j)$ and $\sigma_t^*(\underline{y_t}, e_i, e_j)$ are respectively given by:

$$\mu^*_{i,j,t} = \frac{\mu_{i,j,t} + \sigma^2_{i,j,t}\theta'_{2,1}e_j}{1 - 2\sigma^2_{i,j,t}\theta_{2,2}} \quad \text{and} \quad \sigma^*_{i,j,t} = \left(\frac{\sigma^2_{i,j,t}}{1 - 2\sigma^2_{i,j,t}\theta_{2,2}}\right)^{1/2}, \tag{34}$$

and where:

$$\begin{aligned} &\xi_{t+1}|z_{t+1}, \underline{z_t}, \underline{y_t} \overset{\mathbb{P}^*}{\sim} N(0,1)\,, \\ &\mathbb{P}^*(z_{t+1} = e_j | z_t = e_i, \underline{z_{t-1}}, \underline{y_t}) = \pi^*_{i,j}(\underline{y_t}) = \pi^*_{i,j,t}\,, \quad \forall (e_i, e_j) \in \mathcal{E} \times \mathcal{E}\,. \end{aligned} \tag{35}$$

4.2 Pricing within $CGSR$ Economies

- The investor's information at each date t is the $(J+1)$-dimensional factor $w_t = (y_t, z_t')'$ introduced in the previous section, and the SDF $M_{t,t+1}^{(S)}$ is:

$$M_{t,t+1}^{(S)} = \exp\left[-r_{t+1} + \alpha_{1,t}' z_{t+1} + \alpha_{2,1,t}' z_{t+1} y_{t+1} + \alpha_{2,2,t} y_{t+1}^2 - \widetilde{\psi}_{S,t}(\alpha_{1,t}, \alpha_{2,t})' z_t\right], \tag{36}$$

- where $\widetilde{\psi}_{S,t}(\alpha_{1,t}, \alpha_{2,t}) = \log \widetilde{\varphi}_{S,t}(\alpha_{1,t}, \alpha_{2,t})$, and denoting $\alpha_{2,t} = (\alpha_{2,1,t}', \alpha_{2,2,t})'$.

- Pricing of not only the (conditionally) mean-based and variance-based sources of risk but also the pricing, by means of the risk-correction coefficient $\alpha_{1,t}$ and $\alpha_{2,1,t}$, of regime-shift risk.

- The ICC requires (Direct Modelling setting) the following constraint on model parameters and risk-sensitivity vectors:

$$\left[\widetilde{\psi}_{S,t}(\alpha_{1,t}, \alpha_{2,1,t} + e, \alpha_{2,2,t}) - \widetilde{\psi}_{S,t}(\alpha_{1,t}, \alpha_{2,1,t}, \alpha_{2,2,t})\right]' z_t = r_{t+1}\,, \quad \forall(\underline{y_t}, \underline{z_t})\,, \tag{37}$$

where e is the J-dimensional vector whose components are equal to 1.

- In the Back Modelling case, the ICC become:

$$\widetilde{\psi}^{\mathbb{Q}}_{S,t}(0, e, 0)' z_t = r_{t+1}\,, \tag{38}$$

where $\widetilde{\psi}^{\mathbb{Q}}_{S,t}(u_1, u_{2,1}, u_{2,2}) = \log \widetilde{\varphi}^{\mathbb{Q}}_{S,t}(u_1, u_{2,1}, u_{2,2})$.

4.3 The *Additive* $CGSR$ Economy

- We aim at specifying a Gaussian regime-switching economy characterized by a flexible exponential-quadratic SDF-based change of probability measure and a Car (discrete-time affine) risk-neutral dynamics.

- Let us assume a regime-switching model (27) with:

$$\begin{aligned} \mu(\underline{y_t}, z_t, z_{t+1}) &= \mu_{0,1} + \mu_{0,2} y_t + \mu_1' z_t + \mu_2' z_{t+1}\,, \\ \sigma^2(\underline{y_t}, z_t, z_{t+1}) &= (\sigma_1' z_t + \sigma_2' z_{t+1})^2\,, \end{aligned} \tag{39}$$

$\rightarrow$ then, the specification (39) makes the joint dynamics of $w_t = (y_t, z_t')'$ Car [see Appendix 3 of the paper].

4.4 A $\mathbb{Q}$-Affine $CGSR$ Asset Pricing Model

- Let us present now a *Additive* $CGSR$ asset pricing model able to propose at the same time a tractable (explicit or quasi explicit) pricing formula and an exponential-quadratic SDF-based change of probability measure with non-linear risk-correction coefficients and, thus, a flexible historical dynamics.

- We follow the Back Modelling strategy, and we assume that the $\mathbb{Q}$-dynamics is given by Car specification (39).

- Then we impose the ICC $E_t^{\mathbb{Q}}[\exp(y_{t+1})] = \exp(r_{t+1})$, and we obtain:

$$y_{t+1} = r_{t+1} - \left[\lambda(\mu_2^{\mathbb{Q}}, \sigma_1^{\mathbb{Q}}, \sigma_2^{\mathbb{Q}}, \pi^{\mathbb{Q}}) + \frac{1}{2}(\sigma_1^{\mathbb{Q}})^2\right]' z_t + \mu_2^{\mathbb{Q}\prime} z_{t+1} + (\sigma_1^{\mathbb{Q}\prime} z_t + \sigma_2^{\mathbb{Q}\prime} z_{t+1})\,\xi_{t+1}\,,$$

$$\xi_{t+1}|z_{t+1}, \underline{z_t}, \underline{y_t} \overset{\mathbb{Q}}{\sim} N(0,1)\,,$$

$$\mathbb{Q}(z_{t+1} = e_j | z_t = e_i, \underline{z_{t-1}}, \underline{y_t}) = \mathbb{Q}(z_{t+1} = e_j | z_t = e_i) = \pi_{i,j}^{\mathbb{Q}}\,, \tag{40}$$

where $\lambda_i(\mu_2^{\mathbb{Q}}, \sigma_1^{\mathbb{Q}}, \sigma_2^{\mathbb{Q}}, \pi^{\mathbb{Q}}) = \log \sum_{j=1}^{J} \pi_{i,j}^{\mathbb{Q}} \exp\left(\mu_{2,j}^{\mathbb{Q}} + \frac{1}{2}(\sigma_{2,j}^{\mathbb{Q}})^2 + \sigma_{1,i}^{\mathbb{Q}}\,\sigma_{1,j}^{\mathbb{Q}}\right)$.

- Then, if we specify an exponential-quadratic SDF:

$$M_{t,t+1}^{(S)} = \exp\left[-r_{t+1} + \alpha_{1,t}' z_{t+1} + \alpha_{2,1,t}' z_{t+1} y_{t+1} + \alpha_{2,2,t} y_{t+1}^2 - \widetilde{\psi}_{S,t}(\alpha_{1,t}, \alpha_{2,t})' z_t\right], \tag{41}$$

the historical dynamics is $\mathbb{P}_{t+1} = S_{(-\alpha_{1,t}, -\alpha_{2,t})}(\mathbb{Q}_{t+1})$.

- More precisely:

$$y_{t+1} = \mu_t(\underline{y_t}, z_t, z_{t+1}) + \sigma_t(\underline{y_t}, z_t, z_{t+1})\varepsilon_{t+1}$$
$$\varepsilon_{t+1} \,|\, z_{t+1}, \underline{z_t}, \underline{y_t} \overset{\mathbb{P}}{\sim} N(0,1)\,, \tag{42}$$

- with

$$\mu_t(\underline{y_t}, e_i, e_j) = \frac{r_{t+1} - \left[\lambda_i(\mu_2^{\mathbb{Q}}, \sigma_1^{\mathbb{Q}}, \sigma_2^{\mathbb{Q}}, \pi^{\mathbb{Q}}) + \frac{1}{2}(\sigma_{1,i}^{\mathbb{Q}})^2\right] + \mu_{2,j}^{\mathbb{Q}} - (\sigma_{1,i}^{\mathbb{Q}} + \sigma_{2,j}^{\mathbb{Q}})^2 \alpha'_{2,1,t} e_j}{1 + 2\,(\sigma_{1,i}^{\mathbb{Q}} + \sigma_{2,j}^{\mathbb{Q}})^2\,\alpha_{2,2,t}}$$

$$\sigma_t(\underline{y_t}, e_i, e_j) = \left(\frac{(\sigma_{1,i}^{\mathbb{Q}} + \sigma_{2,j}^{\mathbb{Q}})^2}{1 + 2\,(\sigma_{1,i}^{\mathbb{Q}} + \sigma_{2,j}^{\mathbb{Q}})^2\,\alpha_{2,2,t}}\right)^{1/2}, \tag{43}$$

- where

$$\mathbb{P}(z_{t+1} = e_j | z_t = e_i, \underline{z_{t-1}}, \underline{y_t}) = \pi_{i,j}(\underline{y_t}) = \pi_{i,j,t} \quad \forall (e_i, e_j) \in \mathcal{E} \times \mathcal{E}$$

$$\pi_{i,j,t} = \frac{\pi^{\mathbb{Q}}_{i,j,t} \exp(-\alpha'_{1,t} e_j) \widetilde{\varphi}^{\mathbb{Q}}_{S,t,i,j}(-\alpha'_{2,1,t} e_j, -\alpha_{2,2,t})}{\sum_{j=1}^{J} \pi^{\mathbb{Q}}_{i,j,t} \exp(-\alpha'_{1,t} e_j) \widetilde{\varphi}^{\mathbb{Q}}_{S,t,i,j}(-\alpha'_{2,1,t} e_j, -\alpha_{2,2,t})}. \tag{44}$$

- and where

$$\widetilde{\varphi}^{\mathbb{Q}}_{S,t,i,j}(-\alpha'_{2,1,t} e_j, -\alpha_{2,2,t}) = \exp\left\{ -\frac{1}{2} \log\left[1 + 2\,(\sigma^{\mathbb{Q}}_{1,i} + \sigma^{\mathbb{Q}}_{2,j})^2\,(\alpha'_{2,1,t} e_j) \right] \right.$$

$$-\frac{1}{2} \frac{\left(r_{t+1} - \left[\lambda_i(\mu^{\mathbb{Q}}_2, \sigma^{\mathbb{Q}}_1, \sigma^{\mathbb{Q}}_2, \pi^{\mathbb{Q}}) + \frac{1}{2}(\sigma^{\mathbb{Q}}_{1,i})^2 \right] + \mu^{\mathbb{Q}}_{2,j} \right)^2}{(\sigma^{\mathbb{Q}}_{1,i} + \sigma^{\mathbb{Q}}_{2,j})^2}$$

$$\left. +\frac{1}{2} \frac{\left(r_{t+1} - \left[\lambda_i(\mu^{\mathbb{Q}}_2, \sigma^{\mathbb{Q}}_1, \sigma^{\mathbb{Q}}_2, \pi^{\mathbb{Q}}) + \frac{1}{2}(\sigma^{\mathbb{Q}}_{1,i})^2 \right] + \mu^{\mathbb{Q}}_{2,j} - (\sigma^{\mathbb{Q}}_{1,i} + \sigma^{\mathbb{Q}}_{2,j})^2\,\alpha'_{2,1,t} e_j \right)^2}{(\sigma^{\mathbb{Q}}_{1,i} + \sigma^{\mathbb{Q}}_{2,j})^2 + 2(\sigma^{\mathbb{Q}}_{1,i} + \sigma^{\mathbb{Q}}_{2,j})^4 \alpha_{2,2,t}} \right\}.$$

- We have specified a Gaussian Regime Switching Security Market model with a Car (Affine) $\mathbb{Q}$-dynamics providing explicit or quasi explicit pricing formula

- and characterized by an exponential-quadratic SDF-based change of probability measure, with non-linear stochastic risk-correction coefficients,

- generating a very large set of historical dynamics but still computationally tractable (being conditionally Gaussian).

5. Conclusions and Further Developments

- In this paper we have proposed, working with discrete time no-arbitrage asset pricing models, to widen the bridge between the historical and the risk-neutral factor distribution, while keeping, respectively, flexible and tractable the modelling of both dynamics.

- The key tools behind this more general change of probability measure are the Second-Order Esscher Transform and the Second-Order Laplace Transform. The associated change of probability measure is thus generated by an Exponential-Quadratic Stochastic Discount Factor, specified by first-order and second-order stochastic risk-sensitivity vectors.

- We have shown the large flexibility of this new approach in the case of conditionally Gaussian dynamics and conditionally Gaussian Switching Regime dynamics.

- These classes provide a large variety of security market models.

- In particular, Gaussian switching regime models show several degrees of flexibility, both under the historical and risk-neutral probability, given the serial dependence of regimes and the introduction of the regime indicator function and of the linear and quadratic term in the log-pricing kernel.

- Our approach can be coupled with a Back Modelling strategy assuming a Car risk-neutral factor dynamics and then obtaining an historical dynamics by means of a Second-Order Esscher Transform with risk-sensitivity coefficients specified as any functions of the state vector. In this case we have at the same time explicit or quasi explicit pricing formulas for several derivative assets and a very large set of possible historical dynamics that remain computationally tractable.

- Although we have illustrated our approach using security market models, our results could be applied in many other asset pricing contexts like yield curve and credit risk models.

- We leave these developments to future research.